CHAPTER 7

ADAPTIVE CONTROL OF A CLASS OF NONLINEAR SYSTEMS

7.1 INTRODUCTION

In recent years there has been a great deal of interest in the use of state feedback to exactly linearize the input-output behavior of nonlinear control systems, for example of the form

$$\dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)\, u_i$$
$$y_1 = h_1(x) \ldots y_p = h_p(x) \qquad (7.1.1)$$

In (7.1.1), $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^p$ and $f, g_i : \mathbb{R}^n \to \mathbb{R}^n$. Also the h_j's are scalar valued functions from $\mathbb{R}^n$ to $\mathbb{R}$. The theory of linearization by exact feedback was developed through the efforts of several researchers, such as Singh and Rugh [1972], Freund [1975], Meyer & Cicolani [1980], Isidori, Krener, Gori-Giorgi & Monaco [1981] in the continuous time case. Good surveys are available in Claude [1986], Isidori [1985, 1986] and Byrnes & Isidori [1984]. The discrete time case and sampled data cases are more involved and are developed in Monaco & Normand-Cyrot [1986]. A number of applications of these techniques have been made. Their chief drawback however seems to be in the fact that they rely on exact cancellation of nonlinear terms in order to get linear input-output behavior. Consequently, if there are errors in the model of the nonlinear terms, the cancellation is no longer exact and the input-output behavior no longer linear. In this chapter, we suggest the

use of parameter adaptive control to help make more robust the cancellation of the nonlinear terms when the uncertainty in the nonlinear terms is parametric. The results of this chapter are based on Sastry & Isidori [1987].

The remainder of the chapter is organized as follows: we give a brief review of linearization theory along with the concept of a minimum phase nonlinear system in Section 7.2. We discuss the adaptive version of this control strategy in Section 7.3 along with its application to the adaptive control of rigid robot manipulators, based on Craig, Hsu & Sastry [1987]. In Section 7.4, we collect some suggestions for future work.

7.2 LINEARIZING CONTROL FOR A CLASS OF NONLINEAR SYSTEMS—A REVIEW

7.2.1 Basic Theory

SISO Case

A large class of nonlinear control systems can be made to have linear input-output behavior through a choice of *nonlinear state feedback* control law. Consider, at first, the single-input single-output system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) \qquad (7.2.1)$$

with $x \in \mathbb{R}^n$, f, g, h all smooth nonlinear functions. In this chapter, a *smooth function* will mean an infinitely differentiable function. Differentiating y with respect to time, one obtains

$$\dot{y} = \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x) u$$
$$:= L_f h(x) + L_g h(x) u \qquad (7.2.2)$$

where $L_f h(x) : \mathbb{R}^n \to \mathbb{R}$ and $L_g h(x) : \mathbb{R}^n \to \mathbb{R}$ stand for the *Lie derivatives* of h with respect to f, g respectively. If $L_g h(x)$ is bounded away from zero for all x, the state feedback control law (of the form $u = \alpha(x) + \beta(x)v$)

$$u = \frac{1}{L_g h}(-L_f h + v) \qquad (7.2.3)$$

yields the linear system (linear from the new input v to y)

$$\dot{y} = v$$

The control law (7.2.3) has the effect of rendering $(n-1)$ of the states of

(7.2.1) unobservable through appropriate choice of state feedback.

In the instance that $L_g h(x) \equiv 0$ meaning $L_g h(x) = 0$ for all x, one differentiates (7.2.2) further to get

$$\ddot{y} = L_f^2 h(x) + (L_g L_f h)(x) u \tag{7.2.4}$$

In (7.2.4), $L_f^2 h(x)$ stands for $L_f(L_f h)(x)$ and $L_g L_f h(x) = L_g(L_f h(x))$. As before, if $L_g L_f h(x)$ is bounded away from zero for all x, the control law

$$u = \frac{1}{L_g L_f h(x)} (-L_f^2 h(x) + v) \tag{7.2.5}$$

linearizes the system (7.2.4) to yield

$$\ddot{y} = v$$

More generally, if γ is the smallest integer such that $L_g L_f^i h \equiv 0$ for $i = 0, \ldots, \gamma - 2$ and $L_g L_f^{\gamma - 1} h$ is bounded away from zero, then the control law

$$u = \frac{1}{L_g L_f^{\gamma - 1} h} (-L_f^{\gamma} h + v) \tag{7.2.6}$$

yields

$$y^{\gamma} = v$$

The procedure described above terminates at some finite γ if the row vectors $\{\frac{dh}{dx}(x), \frac{d}{dx} L_f h(x), \ldots, \frac{d}{dx} L_f^{\eta - 1} h(x)\}$ are linearly independent for all x.

Note that the theory is considerably more complicated and incomplete if $L_g L_f^{\gamma - 1} h$ is not identically zero, but is equal to zero for some values of x.

MIMO Case—Static State Feedback

For the multi-input multi-output case, we consider *square* systems (that is systems with as many inputs as outputs) of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x) u_1 + \cdots + g_p(x) u_p \\ y_1 &= h_1(x) \\ &\vdots \\ y_p &= h_p(x) \end{aligned} \tag{7.2.7}$$

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^p$ and f, g_i, h_j, are assumed smooth. Now, differentiate the $j\,th$ output y_j with respect to time to get

$$\dot{y}_j = L_f h_j + \sum_{i=1}^{p} (L_{g_i} h_j) u_i \tag{7.2.8}$$

In (7.2.8), note that if each of the $(L_{g_i} h_j)(x) \equiv 0$, then the inputs do not appear in (7.2.8). Define γ_j to be the smallest integer such that at least one of the inputs appears in $y_j^{\gamma_j}$, that is,

$$y_j^{\gamma_j} = L_f^{\gamma_j} h_j + \sum_{i=1}^{p} L_{g_i}(L_f^{\gamma_j - 1} h_j) u_i \tag{7.2.9}$$

with at least one of the $L_{g_i}(L_f^{\gamma_j - 1} h_j) \neq 0$, for some x. Define the $p \times p$ matrix $A(x)$ as

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1 & \cdots & L_{g_p} L_f^{\gamma_1 - 1} h_1 \\ \vdots & & \vdots \\ L_{g_1} L_f^{\gamma_p - 1} h_p & \cdots & L_{g_p} L_f^{\gamma_p - 1} h_p \end{bmatrix} \tag{7.2.10}$$

Then, (7.2.9) may be written as

$$\begin{bmatrix} y_1^{\gamma_1} \\ \vdots \\ y_p^{\gamma_p} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_p} h_p \end{bmatrix} + A(x) \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \tag{7.2.11}$$

If $A(x) \in \mathbb{R}^{p \times p}$ is bounded away from singularity (meaning that $A^{-1}(x)$ exists for all x and has bounded norm), the state feedback control law

$$u = -A(x)^{-1}\begin{bmatrix} L_f^{\gamma_1}h_1 \\ \vdots \\ L_f^{\gamma_p}h_p \end{bmatrix} + A(x)^{-1}v \tag{7.2.12}$$

yields the *linear* closed loop system

$$\begin{bmatrix} y_1^{\gamma_1} \\ \vdots \\ y_p^{\gamma_p} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \tag{7.2.13}$$

Note that the system of (7.2.13) is in addition decoupled. Thus, decoupling is achieved as a by-product of linearization. A happy consequence of this is that a large number of SISO results are easily extended to this class of MIMO systems. Thus, for example, once linearization has been achieved, any further control objective such as model matching, pole placement or tracking can be easily met. The feedback law (7.2.12) is referred to as a *static state feedback linearizing control law.*

MIMO Case—Dynamic State Feedback

If $A(x)$ as defined in (7.2.10) is singular, and the drift term in (7.2.11) (i.e. the first term on the right-hand side) is not in the range of $A(x)$, linearization may still be achieved by using dynamic state feedback. To keep the notation from proliferating, we review the methods in the case when $p = 2$ (two inputs, two outputs). $A(x)$ then has rank 1 for all x. Using elementary column operations, we may compress $A(x)$ to one column, i.e:,

$$A(x)\,T(x) = \begin{bmatrix} \tilde{a}_{11}(x) & 0 \\ \tilde{a}_{21}(x) & 0 \end{bmatrix}$$

with $T(x) \in \mathbb{R}^{2\times 2}$ a nonsingular matrix. Now, defining the new inputs $w = T^{-1}(x)u$, (7.2.11) reads

$$\begin{bmatrix} y_1^{\gamma_1} \\ y_2^{\gamma_2} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1}h_1 \\ L_f^{\gamma_2}h_2 \end{bmatrix} + \begin{bmatrix} \tilde{a}_{11}(x) \\ \tilde{a}_{21}(x) \end{bmatrix} w_1 \tag{7.2.14}$$

Also, (7.2.7) now reads as

$$\dot{x} = f(x) + \tilde{g}_1(x)w_1 + \tilde{g}_2(x)w_2 \tag{7.2.15}$$

where

$$[\tilde{g}_1(x)\ \tilde{g}_2(x)] = [g_1(x)\ g_2(x)]\,T(x)$$

Differentiating (7.2.14), and using (7.2.15), we get

$$\begin{bmatrix} y_1^{\gamma_1+1} \\ y_2^{\gamma_2+1} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1+1}h_1 + L_{\tilde{g}_1}L_f^{\gamma_1}h_1w_1 + L_f\tilde{a}_{11}w_1 + L_{\tilde{g}_1}\tilde{a}_{11}w_1^2 \\ L_f^{\gamma_2+1}h_2 + L_{\tilde{g}_1}L_f^{\gamma_2}h_2w_1 + L_f\tilde{a}_{21}w_1 + L_{\tilde{g}_1}\tilde{a}_{21}w_1^2 \end{bmatrix}$$
$$+ \begin{bmatrix} \tilde{a}_{11} & L_{\tilde{g}_2}L_f^{\gamma_1}h_1 + L_{\tilde{g}_2}\tilde{a}_{11}w_1 \\ \tilde{a}_{21} & L_{\tilde{g}_2}L_f^{\gamma_2}h_2 + L_{\tilde{g}_2}\tilde{a}_{21}w_1 \end{bmatrix}\begin{bmatrix} \dot{w}_1 \\ w_2 \end{bmatrix}$$

The two large blocks in the equation can be condensed to yield:

$$\begin{bmatrix} y_1^{\gamma_1+1} \\ y_2^{\gamma_2+1} \end{bmatrix} = C(x, w_1) + B(x, w_1)\begin{bmatrix} \dot{w}_1 \\ w_2 \end{bmatrix} \tag{7.2.16}$$

Note the appearance of the control term $\dot{w}_1$. Specifying $\dot{w}_1$ is equivalent to the placement of an integrator before w_1, that is to the addition of dynamics to the controller. Now, note that if $B(x, w_1)$ is bounded away from singularity, then the control law

$$\begin{bmatrix} \dot{w}_1 \\ w_2 \end{bmatrix} = -B^{-1}(x, w_1)C(x, w_1) + B^{-1}(x, w_1)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.2.17}$$

yields the linearized system

$$\begin{bmatrix} y_1^{\gamma_1+1} \\ y_2^{\gamma_2+1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.2.18}$$

The control law (7.2.17) is a *dynamic* state feedback, linearizing, and decoupling control law. If $B(x, w_1)$ is singular, the foregoing procedure may be repeated on $B(x, w_1)$. The procedure ends in finitely many steps if and only if the system is right invertible (for details, see Descusse & Moog [1985]).

7.2.2 Minimum Phase Nonlinear Systems

The linearizing control law (7.2.3) when applied to the system (7.2.1) results in a first order input output system. Consequently, $(n-1)$ of the original n states are rendered unobservable by the state feedback. To see this more clearly, consider the linear case, i.e., $f(x) = Ax$, $g(x) = b$, and $h(x) = c^T x$. Then, the condition

$$L_g h(x) \neq 0 \iff c^T b \neq 0$$

and the control law of (7.2.3) is

$$u = \frac{1}{c^T b}(-c^T A x + v) \qquad (7.2.19)$$

resulting in the closed loop system

$$\dot{x} = \left[I - \frac{bc^T}{c^T b} \right] Ax + \frac{b}{c^T b} v$$
$$y = c^T x \qquad (7.2.20)$$

From the fact that the control law (7.2.19) yields a transfer function of $\frac{1}{s}$ from v to y it follows that $(n-1)$ of the eigenvalues of the closed loop matrix $(I - bc^T / c^T b)A$ are located at the zeros of the original system, and the last at the origin. Thus, the linearizing control laws may be thought of as being the nonlinear counterpart of this specific pole placement control law. The dynamics of the states rendered unobservable are indeed the so-called *zero-dynamics* of the system (see (7.2.23)). Clearly, in order to have internal stability (and boundedness of the states), it is important to have the closed loop pole-zero cancellation be stable, i.e. the system be minimum phase. This motivates the understanding and definitions of minimum phas: nonlinear systems. We start with the single-input single-output case.

7.2.2.1 The Single-Input Single-Output Case

The first definition to be made is that of relative degree (or pole-zero excess).

Definition Strong Relative Degree

The system (7.2.1) is said to have *strong relative degree* γ if

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{\gamma-2} h(x) = 0 \quad \text{for all } x$$

and for all x, $L_g L_f^{\gamma-1} h(x)$ is bounded away from zero.

Comments

a) The system (7.2.1) is said to have strong relative degree γ if the output y needs to be differentiated γ times before terms involving the input appear.

b) In the instance that the system has strong relative degree γ, it is possible to verify that, for each $x^\circ \in \mathbb{R}^n$, there exists a neighborhood U° of x° such that the mapping

$$T : U^\circ \to \mathbb{R}^n$$

defined as

$$T_1(x) = z_{11} = h(x)$$
$$T_2(x) = z_{12} = L_f h(x)$$
$$\vdots$$
$$T_\gamma(x) = z_{1\gamma} = L_f^{\gamma-1} h(x) \qquad (7.2.21)$$

and $T_{\gamma+1}, \ldots, T_n$ chosen such that

$$\frac{\partial}{\partial x}\left[T_i(x) \right] g(x) = 0 \quad \text{for } i = \gamma+1, \ldots, n$$

is a diffeomorphism onto its image (see Isidori [1986]).

If we denote by $z_1 \in \mathbb{R}^\gamma$ the vector $(z_{11}, \ldots, z_{1\gamma})^T$ and by $z_2 \in \mathbb{R}^{n-\gamma}$ the vector $(T_{\gamma+1}, \ldots, T_n)^T$, it follows that the equations (7.2.1) may be replaced by

$$\dot{z}_{11} = z_{12}$$
$$\vdots$$
$$\dot{z}_{1\gamma-1} = z_{1\gamma}$$
$$\dot{z}_{1\gamma} = f_1(z_1, z_2) + g_1(z_1, z_2) u$$
$$\dot{z}_2 = \Psi(z_1, z_2)$$

$$y = z_{11} \tag{7.2.22}$$

In the equations above, $f_1(z_1, z_2)$ represents $L_f^{\gamma} h(x)$ and $g_1(z_1, z_2)$ represents $L_g L_f^{\gamma-1} h(x)$ in the new coordinates. $\Psi_i(z_1, z_2)$ represents $L_f T_i(x)$ for $i = \gamma + 1, \ldots, n$. Also note that the input does not directly influence the z_2 states. The representation of the system (7.2.1) through (7.2.22) is called a *normal form*.

If $x = 0$ is an equilibrium point of the undriven system, that is, $f(0) = 0$ and $h(0) = 0$ (without loss of generality), then the dynamics

$$\dot{z}_2 = \Psi(0, z_2) \tag{7.2.23}$$

are referred to as the *zero-dynamics*. Note that the subset

$$L_0 = \{x \in U^\circ \mid h(x) = \cdots = L_f^{\gamma-1}h(x) = 0\}$$

can be made invariant by choosing

$$u = \frac{1}{g_1(z_1, z_2)} \left[-f_1(z_1, z_2) + v \right] \tag{7.2.24}$$

The dynamics of (7.2.23) are the dynamics on this subspace.

Definition Minimum Phase

The nonlinear system (7.2.1) is said to be *globally (locally) minimum phase* if the zero-dynamics are globally (locally) asymptotically stable.

Comments

a) This definition may be strengthened to exponentially stable, in which case we call the system exponentially minimum phase.

b) The previous analysis identifies the normal form (7.2.22) and the zero-dynamics of (7.2.23) only *locally* around any point x° of $\mathbb{R}^n$. Recent work of Byrnes & Isidori [1988] has identified necessary and sufficient conditions for the existence of a globally defined normal form. They have shown that a global version of the notion of zero-dynamics is that of a dynamical system evolving on the smooth manifold of $\mathbb{R}^n$

$$L_0 = \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \cdots = L_f^{\gamma-1}h(x) = 0\}$$

and hereby defined the vector field

$$\bar{f}(x) = f(x) - \frac{L_f^{\gamma} h(x)}{L_g L_f^{\gamma-1} h(x)} g(x) \qquad x \in L_0$$

Note that this is a vector field on L_0 because $\bar{f}(x)$ is tangent to L_0. If

L_0 is connected and the zero-dynamics are globally asymptotically stable (i.e. if the system is *globally minimum phase*), then the normal forms of (7.2.22) are globally defined if and only if the vector fields

$$\bar{g}(x),\ ad_{\bar{f}}\bar{g}(x), \ldots,\ ad_{\bar{f}}^{\gamma-1}\bar{g}(x)$$

are complete (i.e. have no finite escape time), where

$$\bar{g}(x) = \frac{1}{L_g L_f^{\gamma-1} h(x)} g(x) \qquad \bar{f}(x)\ , \quad \text{as above}$$

while

$$ad_{\bar{f}}\bar{g} = \frac{\partial \bar{g}}{\partial x} \bar{f}(x) - \frac{\partial \bar{f}}{\partial x} \bar{g}(x)$$

is the so-called *Lie bracket* of $\bar{f}$, $\bar{g}$ and $ad_{\bar{f}}^i \bar{g} = ad_{\bar{f}} \ldots ad_{\bar{f}} \bar{g}$ iterated i times. This is in turn guaranteed by requiring that the vector fields in question be globally Lipschitz continuous, for example.

The utility of the definition of minimum phase zero-dynamics arises in the context of tracking: if the control objective is for the output $y(t)$ to track a pre-specified reference trajectory $y_m(t)$, then the control input

$$v = y_m^{\gamma} + \alpha_{\gamma}(y_m^{\gamma-1} - y^{\gamma-1}) + \cdots + \alpha_1(y_m - y) \tag{7.2.25}$$

results in the following equation for the tracking error $e_0 := y - y_m$

$$e_0^{\gamma} + \alpha_{\gamma} e_0^{\gamma-1} + \cdots + \alpha_1 e_0 = 0 \tag{7.2.26}$$

It is important to note that the control law of (7.2.25) is not implemented by differentiating y repeatedly but rather as a *state feedback law* since

$$\dot{y} = L_f h$$

$$\ddot{y} = L_f^2 h$$

$$\vdots$$

$$y^{\gamma-1} = L_f^{\gamma-1} h$$

If $\alpha_1, \ldots, \alpha_\gamma$ are chosen so that $s^\gamma + \alpha_\gamma s^{\gamma-1} + \cdots + \alpha_1$ is a Hurwitz polynomial, then it is easy to see that $e_0, \dot{e}_0, \ldots, e_0^{\gamma-1}$ go to zero as t tends to ∞. Further, if $y_m, \dot{y}_m, \ldots, y_m^{\gamma-1}$ are bounded, then $y, \dot{y}, \ldots, y^{\gamma-1}$ are also bounded and so is z_1. The following proposition guarantees bounded tracking, that is tracking with bounded states.

Proposition 7.2.1 Bounded Tracking in Minimum Phase Nonlinear Systems

If the zero-dynamics of the nonlinear system (7.2.1) as defined in (7.2.23) are globally exponentially stable. Further, $\Psi(z_1, z_2)$ in (7.2.22) has continuous and bounded partial derivatives in z_1, z_2 and $y_m, \dot{y}_m, \ldots, y_m^{\gamma-1}$ are bounded

Then the control law (7.2.24)–(7.2.25) results in bounded tracking, that is, $x \in \mathbb{R}^n$ is bounded and $y(t)$ converges to $y_m(t)$.

Proof of Proposition 7.2.1

From the foregoing discussion, it only remains to show that z_2 is bounded. We accomplish this by using the converse theorem of Lyapunov of theorem 1.5.1 (as we did for theorem 5.3.1).

Since (7.2.23) is (globally) exponentially stable and Ψ has bounded derivatives, there exists $v_1(z_2)$ such that

$$\alpha_1 |z_2|^2 \le v_1(z_2) \le \alpha_2 |z_2|^2$$

$$\frac{dv_1}{dz_2}\Psi(0, z_2) \le -\alpha_3 |z_2|^2$$

$$\left|\frac{dv_1}{dz_2}\right| \le \alpha_4 |z_2| \tag{7.2.27}$$

By assumption, the control law (7.2.24)–(7.2.25) yields bounded z_1, i.e.

$$|z_1(t)| \le k \qquad \text{for all } t \tag{7.2.28}$$

Using (7.2.27) for the system (7.2.22) yields

$$\dot{v}_1(t) = \frac{dv_1}{dz_2}\Psi(z_1, z_2) \le -\alpha_3 |z_2|^2 + \frac{dv_1}{dz_2}(\Psi(z_1, z_2) - \Psi(0, z_2))$$

$$\le -\alpha_3 |z_2|^2 + \alpha_4 kl |z_2| \tag{7.2.29}$$

where l is the Lipschitz constant of $\Psi(z_1, z_2)$ in z_1 (Ψ is globally Lipschitz since it has bounded partial derivatives). It is now easy to see

that

$$\dot{v}_1 \le 0 \qquad \text{for} \qquad |z_2| \ge \left[\frac{\alpha_4 kl}{\alpha_3}\right]$$

Using this along with the bounds in (7.2.27), it is easy to establish that z_2 is bounded. □

Comments

a) Proposition 7.2.1 is a global proposition. If the zero-dynamics were only locally exponentially stable, the proposition would yield that z_2 is bounded for small enough z_1, that is, for small enough $y_m, \dot{y}_m, \ldots, y_m^{\gamma-1}$.

b) The assumptions of proposition 7.2.1 call for a strong form of stability—exponential stability. In fact, counter-examples to the proposition exist if the zero-dynamics are not exponentially stable—for example, if some of the eigenvalues of $d\Psi(0, z_2)/dz_2$ evaluated at $z_2 = 0$ lie on the $j\omega$-axis.

c) However, the hypothesis of proposition 7.2.1 can be weakened considerably without affecting the conclusion. In particular, it is sufficient to ask only that the zero-dynamics of (7.2.23) converge asymptotically to a bounded set (a form of exponential attractivity). To be concrete, the exponential minimum phase hypothesis can be replaced by the condition

$$z_2^T \Psi(0, z_2) \le -\alpha_3 |z_2|^2 \qquad \text{for} \qquad |z_2| \ge k \tag{7.2.30}$$

for some k (large). Condition (7.2.30) is similar to (7.2.27) for the Lyapunov function $|z_2|^2$, except that it holds outside a ball of radius k. It is then easy to verify that all trajectories of the undriven zero-dynamics (7.2.23) eventually converge to a ball and that the proof of proposition 7.2.1 can be repeated to yield bounded tracking. This remark is especially useful in the adaptive context where the assumption of minimum phase zero-dynamics may be replaced by exponential attractivity.

7.2.2.2 The Multi-Input Multi-Output Case

Definitions of zero-dynamics for the square multi-input multi-output case are more subtle, as pointed out in Isidori & Moog [1987]. There are three different ways of defining them, depending on which definition of the zeros of an LTI system one chooses to generalize

a) the dynamics of the maximal controlled invariant manifold in the kernel of the output map, or

b) the output constrained dynamics (with output constrained to zero), or

c) the dynamics of the inverse system.

It is also pointed out that the three different definitions coincide if the nonlinear system can be decoupled by static state feedback, in which case the definition parallels the development of the SISO case above. More specifically, if $A(x)$ as defined in (7.2.10) is non-singular, then we proceed as follows. Define

$$\gamma_1 + \cdots + \gamma_p = m$$

and $z_1 \in \mathbb{R}^m$ by

$$z_1^T = (h_1, L_f h_1, \ldots, L_f^{\gamma_1 - 1} h_1, h_2, \ldots, L_f^{\gamma_2 - 1} h_2, \ldots, h_p, \ldots, L_f^{\gamma_p - 1} h_p)$$

Also, define $z_2 \in \mathbb{R}^{n-m}$ by

$$z_{21} = T_1(x), \ldots, z_{2(n-m)} = T_{n-m}(x)$$

with $z^T = (z_1^T, z_2^T)$ representing a diffeomorphism of the state variables x. In these coordinates, the equations (7.2.1) read as

$$\begin{aligned}
\dot z_{11} &= z_{12} \\
&\;\;\vdots \\
\dot z_{1\gamma_1} &= f_1(z_1, z_2) + g_1(z_1, z_2) u \\
\dot z_{1(\gamma_1+1)} &= z_{1(\gamma_1+2)} \\
&\;\;\vdots \\
\dot z_{1m} &= f_p(z_1, z_2) + g_p(z_1, z_2) u \\
\dot z_2 &= \Psi(z_1, z_2) + \Phi(z_1, z_2) u \\
y_1 &= z_{11} \\
y_2 &= z_{1(\gamma_1+1)} \\
&\;\;\vdots \\
y_p &= z_{1(m-\gamma_p+1)}
\end{aligned} \qquad (7.2.31)\text{–}(7.2.32)$$

Above, $f_1(z_1, z_2)$ represents $L_f^{\gamma_1} h_1(x)$ and $g_1(z_1, z_2)$ the first row of $A(x)$ in the (z_1, z_2) coordinates. The zero-dynamics are defined as follows. Let u be a linearizing control law, for example

$$u(z_1, z_2) = -\begin{bmatrix} g_1(z_1, z_2) \\ \vdots \\ g_p(z_1, z_2) \end{bmatrix}^{-1} \begin{bmatrix} f_1(z_1, z_2) \\ \vdots \\ f_p(z_1, z_2) \end{bmatrix} \qquad (7.2.33)$$

Then, if $0 \in \mathbb{R}^n$ is an equilibrium point of the undriven system, that is $f(0) = 0$ and $h_1(0) = \cdots = h_p(0) = 0$, the zero dynamics are the dynamics of

$$\dot z_2 = \Psi(0, z_2) + \Phi(0, z_2) u(0, z_2) \qquad (7.2.34)$$

It is verified in Isidori & Moog [1987] that the dynamics of (7.2.34) are independent of the choice of linearizing feedback law. Proposition 7.2.1 and the remarks following it can be verified to hold with the hypothesis being on the zero-dynamics of (7.2.34).

In the instance that the system (7.2.1) is not decouplable by static state feedback, the definition of the zero-dynamics is considerably more involved. We do not discuss it here since we will not use it.

7.2.3 Model Reference Control for Nonlinear Systems

The discussion thus far has been restricted to tracking control of linearizable nonlinear systems. For this class of systems the extension to model reference adaptive control is easy: for the single-input single-output case, consider $y_m(t)$ to be the output of a linear time invariant reference model with input $r(t)$, specified by

$$\dot x_m = A_m x_m + b_m r$$
$$y_m = c_m^T x_m \qquad (7.2.35)$$

Then, provided that the relative degree of the reference model is *greater than or equal* to the relative degree γ of the nonlinear system, the control law (7.2.24)–(7.2.25) is easily modified to

$$u = \frac{1}{L_g L_f^{\gamma-1} h}\left[-L_f^{\gamma} h + y_m^{\gamma} + \sum_{i=0}^{\gamma-1} \alpha_{i+1}\left[y_m^i - y^i \right]\right]$$

$$= \frac{1}{L_g L_f^{\gamma-1} h}\left[-L_f^{\gamma} h + c_m^T A_m^{\gamma} x_m + c_m^T A_m^{\gamma-1} b_m r + \sum_{i=0}^{\gamma-1} \alpha_{i+1}\left[c_m^T A_m^i x_m - L_f^i h \right]\right]$$

Note that the dimensions of the model play no role. This also relates to the fact that the tracking error $e_0 := y - y_m$ satisfies the equation

$$e_0^{\gamma} + \alpha_{\gamma} e_0^{\gamma-1} + \cdots + \alpha_1 e_0 = 0 \tag{7.2.36}$$

For the multi-input multi-output case, and the model of the form

$$\dot{x}_m = A_m x_m + B_m r$$
$$y_m = C_m x_m$$

with $B_m \in \mathbb{R}^{n_m \times p}$, $C_m \in \mathbb{R}^{p \times n_m}$, the class of models that can be matched is that for which y_{m1} has relative degree γ_1, y_{m2} has relative degree γ_2 and so on. As above, the model error $e_0 := y - y_m$ satisfies

$$\hat{M}(s) e_0 = 0 \tag{7.2.37}$$

where

$$\hat{M}(s) = \text{diag}\left[\frac{1}{s^{\gamma_1} + \alpha_{1\gamma_1} s^{\gamma_1 - 1} + \cdots + \alpha_{11}}, \ldots, \frac{1}{s^{\gamma_p} + \cdots + \alpha_{p1}}\right] \tag{7.2.38}$$

This is not unlike the results of Section 6.3, where linear multivariable plants are found to match their diagonal Hermite forms.

In the adaptive control sequel to this section, we will consider the tracking scenarios for compactness. Also, as we have noted above, the dimensions of the reference model and its parameters do not play much of a role except to generate $\dot{y}_m, \ddot{y}_m, \ldots, y_m^{\gamma}$.

7.3 ADAPTIVE CONTROL OF LINEARIZABLE MINIMUM PHASE SYSTEMS

In practical implementations of exactly linearizing control laws, the chief drawback is that they are based on exact cancellation of nonlinear terms. If there is any uncertainty in the knowledge of the nonlinear functions f and g, the cancellation is not exact and the resulting input-output equation is not linear. We discuss the use of parameter adaptive control to get asymptotically exact cancellation. At the outset, we will assume that h is known exactly but we will discuss how to relax this assumption later.

7.3.1 Single-Input Single-Output, Relative Degree One Case

Consider a nonlinear system of the form (7.2.1) with $L_g h(x)$ bounded away from zero. Further, let $f(x)$ and $g(x)$ have the form

$$f(x) = \sum_{i=1}^{n_1} \theta_i^{(1)*} f_i(x) \tag{7.3.1}$$

$$g(x) = \sum_{j=1}^{n_2} \theta_j^{(2)*} g_j(x) \tag{7.3.2}$$

where $\theta_i^{(1)*}$, $i = 1, \ldots, n_1$; $\theta_j^{(2)*}$, $j = 1, \ldots, n_2$ are unknown parameters and $f_i(x)$, $g_j(x)$ are *known* functions. At time t, our estimates of the functions f and g are

$$f_e(x) = \sum_{i=1}^{n_1} \theta_i^{(1)}(t) f_i(x) \tag{7.3.3}$$

$$g_e(x) = \sum_{j=1}^{n_2} \theta_j^{(2)}(t) g_j(x) \tag{7.3.4}$$

Here the subscript e stands for estimate and $\theta_i^{(1)}(t)$, $\theta_j^{(2)}(t)$ stand for the estimates of the parameters $\theta_i^{(1)*}$, $\theta_j^{(2)*}$ respectively at time t. Consequently, the linearizing control law (7.2.3) is replaced by

$$u = \frac{1}{(L_g h)_e}\left[-(L_f h)_e + v \right] \tag{7.3.5}$$

with $(L_f h)_e$, $(L_g h)_e$ representing the estimates of $L_f h$, $L_g h$ respectively based on (7.3.3), (7.3.4), i.e.,

$$(L_f h)_e = \sum_{i=1}^{n_1} \theta_i^{(1)}(t) L_{f_i} h \tag{7.3.6}$$

$$(L_g h)_e = \sum_{j=1}^{n_2} \theta_j^{(2)}(t) L_{g_j} h \qquad (7.3.7)$$

If we define $\theta^* \in \mathbb{R}^{n_1+n_2}$ to be the nominal parameter vector $(\theta^{(1)^*}, \theta^{(2)^*})$, $\theta(t) \in \mathbb{R}^{n_1+n_2}$ the parameter estimate, and $\phi = \theta - \theta^*$ the parameter error, then using (7.3.5) in (7.2.2) yields, after some calculation

$$\dot{y} = v + \phi^{(1)^T} w^{(1)} + \phi^{(2)^T} w^{(2)} \qquad (7.3.8)$$

with

$$w^{(1)} \in \mathbb{R}^{n_1} := - \begin{bmatrix} L_{f_1} h \\ \cdot \\ \cdot \\ L_{f_{n_1}} h \end{bmatrix} \qquad (7.3.9)$$

and

$$w^{(2)} \in \mathbb{R}^{n_2} := \begin{bmatrix} L_{g_1} h \\ \cdot \\ \cdot \\ \cdot \\ L_{g_{n_2}} h \end{bmatrix} \frac{((L_f h)_e - v)}{(L_g h)_e} \qquad (7.3.10)$$

The control law for tracking is

$$v = \dot{y}_m + \alpha (y_m - y)$$

and yields the following error equation relating the tracking error $e_0 := y - y_m$ to the parameter error $\phi^T = (\phi^{(1)^T} \phi^{(2)^T})^T$

$$\dot{e}_0 + \alpha e_0 = \phi^T w \qquad (7.3.11)$$

where $w \in \mathbb{R}^{n_1+n_2}$ is defined to be the concatenation of w_1, w_2. Equation (7.3.11) may be written

$$e_0 = \frac{1}{s+\alpha} (\phi^T w)$$

which is of the form of the SPR error equation encountered in Chapter 2. The following theorem may now be stated.

Theorem 7.3.1 Adaptive Tracking
Consider an exponentially minimum phase, nonlinear system of the form (7.2.1), with the assumptions on f, g as given in (7.3.3), (7.3.4). Define the control law

$$u = \frac{1}{(L_g h)_e} \left[-(L_f h)_e + \dot{y}_m + \alpha (y_m - y) \right] \qquad (7.3.12)$$

If $(L_g h)_e$ as defined in (7.3.7) is bounded away from zero and y_m is bounded.

Then the gradient type parameter update law

$$\dot{\phi} = - e_0 w \qquad (7.3.13)$$

yields bounded $y(t)$, asymptotically converging to $y_m(t)$. Further, all state variables of (7.2.1) are bounded.

Proof of Theorem 7.3.1

The Lyapunov function $v(e_0, \phi) = \frac{1}{2} e_0^2 + \frac{1}{2} \phi^T \phi$ is decreasing along the trajectories of (7.3.11), (7.3.13), with $\dot{v}(e_0, \phi) = -\alpha e_0^2 \le 0$. Therefore, e_0 and ϕ are bounded, and $e_0 \in L_2$. To establish that e_0 is uniformly continuous (to use Barbalat's lemma—lemma 1.2.1), or alternately that $\dot{e}_0$ is bounded, we need w—a continuous function of x (since $(L_g h)_e$ is bounded away from zero)—to be bounded. Now note that given a bounded e_0, y_m bounded implies y bounded. From this, and the exponentially minimum phase assumption (proposition 7.2.1), it follows that x is bounded. Hence w is bounded and e_0 is uniformly continuous, and so e_0 tends to zero as $t \to \infty$. □

Comments

a) The preceding theorem guarantees that e_0 converges to zero as $t \to \infty$. Nothing whatsoever is guaranteed about parameter convergence. It is, however, easy to see that both e_0, ϕ converge exponentially to zero if w is persistently exciting, i.e. if there exist $\alpha_1, \alpha_2, \delta > 0$ such that

$$\alpha_2 I \ge \int_{t_0}^{t_0+\delta} w w^T dt \ge \alpha_1 I \qquad (7.3.14)$$

Unfortunately, the condition (7.3.14) is usually impossible to verify explicitly ahead of time, since w is a complicated nonlinear function of x.

b) One other popular way of dealing with parametric uncertainty is to replace the control law in (7.3.12) by the "sliding mode" control law

$$u = \frac{1}{(L_g h)_e} \{ -(L_f h)_e + \dot{y}_m + k \operatorname{sgn}(y_m - y) \} \qquad (7.3.15)$$

The error equation (7.3.11) is then replaced by

$$\dot{e} + k\ \mathrm{sgn}\, e = d(t) \tag{7.3.16}$$

where $d(t)$ is a mismatch term (depending on the difference between $L_g h$ and $(L_g h)_e$, $L_f h$ and $(L_f h)_e, \ldots$). This may be bounded using bounds on f_i, g_j and the ϕ_i's above. It is then possible to see that if $k > \sup_t |d(t)|$, the error e_0 goes to zero, in fact in finite time. This philosophy is not at odds with adaptation as discussed in theorem 7.3.1. In fact, it could be used quite gainfully when the parameter error $\phi(t)$ is small. However, if $\phi(t)$ is large, the gain k needs to be large, resulting in unacceptable chatter, large control activity and other undesirable behavior.

c) An hypothesis of theorem 7.3.1 is that $(L_g h)_e$ be bounded away from zero for all x. Since $(L_g h)_e$ as defined by (7.3.7) may indeed go through zero, even if the 'true' $L_g h$ is bounded away from zero, auxiliary techniques need to be used to guarantee that $(L_g h)_e$ is bounded away from zero. One popular technique is the projection technique, in which the parameters $\theta_1^{(2)}(t), \ldots, \theta_{n_2}^{(2)}(t)$ are kept in a certain parameter range which guarantees that $(L_g h)_e$ is bounded away from zero, say by ϵ (by modifying the update law (7.3.13) as discussed in Chapter 2 and Chapter 3).

7.3.2 Extensions to Higher Relative Degree SISO Systems

We first consider the extensions of the results of the previous section to SISO systems with relative degree γ, that is, $L_g h = L_g L_f h = \cdots = L_g L_f^{\gamma-2} h \equiv 0$ with $L_g L_f^{\gamma-1} h$ bounded away from zero. The non-adaptive linearizing control law is then of the form

$$u = \frac{1}{L_g L_f^{\gamma-1} h}(-L_f^{\gamma} h + v) \tag{7.3.17}$$

If f and g are not completely known but of the form (7.3.1), (7.3.2), we need to replace $L_f^{\gamma} h$ and $L_g L_f^{\gamma-1} h$ by their estimates. We define these as follows

$$(L_f^{\gamma} h)_e := L_{f_e}^{\gamma} h \tag{7.3.18}$$

$$(L_g L_f^{\gamma-1} h)_e := L_{g_e} L_{f_e}^{\gamma-1} h \tag{7.3.19}$$

Note that for $\gamma \geq 2$, (7.3.18), (7.3.19) *are not linear in the unknown parameters* θ_i. For example,

$$(L_f^2 h)_e = \sum_{i=1}^{n_1}\sum_{j=1}^{n_1} L_{f_i}(L_{f_j} h)\theta_i^{(1)}\theta_j^{(1)} \tag{7.3.20}$$

and

$$(L_g L_f h)_e = \sum_{i=1}^{n_2}\sum_{j=1}^{n_1} L_{g_i}(L_{f_j} h)\theta_i^{(2)}\theta_j^{(1)} \tag{7.3.21}$$

and so on.

The development of the preceding section can easily be repeated if we define each of the parameter products to be a new parameter, in which case the $\theta_i^{(1)}\theta_j^{(1)}$ and $\theta_i^{(2)}\theta_j^{(1)}$ of (7.3.20) and (7.3.21) are parameters. Let $\Theta \in \mathbb{R}^k$ be the k-(large!) dimensional vector of parameters $\theta_i^{(1)}$, $\theta_j^{(2)}$, $\theta_i^{(1)}\theta_j^{(2)}$, $\theta_i^{(1)}\theta_j^{(1)}, \ldots$. Thus, for example, if $\gamma = 3$, Θ contains $\theta_i^{(1)}$, $\theta_j^{(2)}$, $\theta_i^{(1)}\theta_j^{(1)}$, $\theta_i^{(1)}\theta_j^{(1)}\theta_k^{(1)}$, $\theta_i^{(1)}\theta_j^{(2)}$, $\theta_i^{(1)}\theta_j^{(1)}\theta_k^{(2)}$. For the purpose of tracking, the control law to be implemented is

$$v = y_m^{\gamma} + \alpha_{\gamma}(y_m^{\gamma-1} - y^{\gamma-1}) + \cdots + \alpha_1(y_m - y)$$

where $\dot{y}$, $\ddot{y}$ are obtained as *state feedback terms* using $\dot{y} = L_f h(x)$, $\ddot{y} = L_f^2 h(x)$, and so on. In the absence of precise information about $L_f h, L_f^2 h, \ldots,$ the tracking law to be implemented is

$$v_e = y_m^{\gamma} + \alpha_{\gamma}\left[y_m^{\gamma-1} - (L_f^{\gamma-1} h)_e\right] + \cdots + \alpha_1(y_m - y) \tag{7.3.22}$$

The adaptive control law then is

$$u = \frac{1}{(L_g L_f^{\gamma-1} h)_e}(-(L_f^{\gamma} h)_e + v_e) \tag{7.3.23}$$

This yields the error equation (with $\Phi := \Theta(t) - \Theta$ representing the parameter error)

$$e_0^{\gamma} + \alpha_{\gamma} e_0^{\gamma-1} + \cdots + \alpha_1 e_0$$

$$= L_f^{\gamma} h + \frac{L_g L_f^{\gamma-1} h}{(L_g L_f^{\gamma-1} h)_e}(-(L_f^{\gamma} h)_e + v_e) - v$$

$$= (L_f^{\gamma} h - (L_f^{\gamma} h)_e)$$

$$+ \left[L_g L_f^{\gamma-1} h - (L_g L_f^{\gamma-1} h)_e \right] \frac{(-(L_f^{\gamma} h)_e + v_e)}{(L_g L_f^{\gamma-1} h)_e} + v_e - v$$

$$:= \Phi^T w_1 + \Phi^T w_2 \qquad (7.3.24)$$

The two terms on the right-hand side arise, respectively, from the mismatch between the ideal and actual linearizing law, and the mismatch between the ideal tracking control v and the actual tracking control v_e. For definiteness, consider the case that $\gamma = 2$ and $n_1 = n_2 = 1$. Then, with $\Theta^T = [\theta^{(1)}, \theta^{(2)}, \theta^{(1)}\theta^{(1)}, \theta^{(1)}\theta^{(2)}]$, we get

$$w_1^T = -\left[0 \quad 0 \quad L_{f_1}^2 h \quad L_{g_1} L_{f_1} h \frac{(-(L_f^2 h)_e + v_e)}{(L_g L_f h)_e} \right]$$

$$w_2^T = -[\, \alpha_1 L_{f_1} h \quad 0 \quad 0 \quad 0 \,] \qquad (7.3.25)$$

Note that w_1 and w_2 can be added to get a regressor w. It is of interest to note that $\theta^{(2)}$ cannot be explicitly identified in this case, since the terms in the regressor multiplying it are zero. Also note that w is a function of x and also $\Theta(t)$.

Consider now the form (7.3.24) of the error equation. For the purposes of adaptation, we could use an error of the form

$$e_1 = \beta_\gamma e_0^{\gamma-1} + \cdots + \beta_1 e_0 \qquad (7.3.26)$$

with the transfer function

$$\frac{\beta_\gamma s^{\gamma-1} + \cdots + \beta_1}{s^\gamma + \alpha_\gamma s^{\gamma-1} + \cdots + \alpha_1} \qquad (7.3.27)$$

strictly positive real. Indeed, if such a signal e_1 were measurable, the basic tracking theorem would follow easily. The difficulty with constructing the signal in (7.3.26) is that $\dot e_0, \ddot e_0, \ldots, e_0^{\gamma-1}$ are not measurable since

$$\dot e_0 = L_f h - \dot y_m$$

$$\ddot e_0 = L_f^2 h - \ddot y_m \qquad (7.3.28)$$

and so on, with $L_f^i h$ not explicitly available since they may not be known functions of x. An exception is a large class of electromechanical systems of which the robot manipulator equations are a special case, see Section 7.3.3 below. In such systems, $\beta_1, \ldots, \beta_\gamma$ may be chosen so that the transfer function in (7.3.27) is strictly positive real and the adaptive

law of the previous section with e_1 as given by (7.3.26) yields the desired conclusion. When the $L_f^i h$'s are not available, the following approach may be used.

Adaptive Control Using an Augmented Error

Motivated by the adaptive schemes of Chapter 3, we define the polynomial

$$\hat L(s) := s^\gamma + \alpha_\gamma s^{\gamma-1} + \cdots + \alpha_1 \qquad (7.3.29)$$

so that equation (7.3.24) may be written as

$$e_0 = \hat L^{-1}(s)(\Phi^T w) \qquad (7.3.30)$$

where we used the hybrid notation of previous chapters and dropped the exponentially decaying initial condition terms. Define the augmented error

$$e_1 = e_0 + \left[\Theta^T \hat L^{-1}(s)(w) - \hat L^{-1}(s)(\Theta^T w) \right] \qquad (7.3.31)$$

Using the fact that constants commute with $\hat L^{-1}(s)$, we get

$$e_1 = e_0 + \left[\Phi^T \hat L^{-1}(s)(w) - \hat L^{-1}(s)(\Phi^T w) \right] \qquad (7.3.32)$$

Note that e_1 in (7.3.31) can be obtained from available signals, unlike (7.3.32) which is used for the analysis. Using (7.3.30) in (7.3.32), we have that

$$e_1 = \Phi^T \hat L^{-1}(s)(w) \qquad (7.3.33)$$

Equation (7.3.33) is a linear error equation. For convenience, we will denote

$$\xi := \hat L^{-1}(s)(w) \qquad (7.3.34)$$

From the error equation (7.3.33), several parameter update laws are immediately suggested. For example, the normalized gradient type algorithm:

$$\dot\Theta = \dot\Phi = \frac{-e_1 \xi}{1 + \xi^T \xi} \qquad (7.3.35)$$

As in the stability proofs of Chapter 3, we will use the following notation

(a) β is a *generic* $L_2 \cap L_\infty$ function which goes to zero as $t \to \infty$.

(b) γ is a *generic* $L_2 \cap L_\infty$ function.

(c) $\| z \|_t$ refers to the norm $\sup_{\tau \le t} | z(\tau)|$, that is the *truncated* L_∞ norm.

From the results of Chapter 2, a number of properties of $\dot{\Phi}$, e_1 follow immediately, with no assumptions on the boundedness of ξ.

Proposition 7.3.2 Properties of the Identifier

Consider the error equation

$$e_1 = \Phi^T \xi \tag{7.3.36}$$

with the update law

$$\dot{\Phi} = \frac{-e_1 \xi}{1 + \xi^T \xi} \tag{7.3.37}$$

Then $\Phi \in L_\infty$, $\dot{\Phi} \in L_2 \cap L_\infty$ and

$$| \Phi^T \xi(t)| \le \gamma(1 + \| \xi \|_t) \qquad \text{for all } t \tag{7.3.38}$$

for some $\gamma \in L_2 \cap L_\infty$.

Proof of Proposition 7.3.2: See theorem 2.4.2.

We are now ready to state and prove the main theorem.

Theorem 7.3.3 Basic Tracking Theorem for SISO Systems with Relative Degree Greater than 1

Consider the control law of (7.3.22)–(7.3.23) applied to an exponentially minimum phase nonlinear system with parameter uncertainty as given in (7.3.1)–(7.3.2).

If $y_m, \dot{y}_m, \ldots, y_m^{\gamma - 1}$ are bounded,
$(L_g L_f^{\gamma - 1} h)_e$ is bounded away from zero,
$f, g, h, L_f^k h, L_g L_f^k h$ are Lipschitz continuous functions, and $w(x, \theta)$ has bounded derivatives in x, θ.

Then the parameter update law

$$\dot{\Phi} = \frac{-e_1 \xi}{1 + \xi^T \xi} \tag{7.3.39}$$

with $\xi = \hat{L}^{-1}(s)(w)$ yields bounded tracking, i.e., $y \to y_m$ as $t \to \infty$ and x is bounded.

Remark: The proof is similar to the proof of theorem 3.7.1 in the linear case. Although the scheme is based on the output error e_0, the choice $\hat{L}^{-1} = \hat{M}$ makes it identical to the input error scheme.

Proof of Theorem 7.3.3

(a) Bounds on the Error Augmentation

Using the swapping lemma (lemma 3.6.5), we have (with notation borrowed from the lemma)

$$\Phi^T \hat{L}^{-1}(w) - \hat{L}^{-1}(\Phi^T w) = -\hat{L}_c^{-1}(\hat{L}_b^{-1}(w^T)\dot{\Phi}) \tag{7.3.40}$$

Using the fact that $\dot{\Phi} \in L_2$ and $\hat{L}_b^{-1}$ is stable (since $\hat{L}^{-1}$ is), we get

$$| (L_b^{-1} w^T) \dot{\Phi} | \le \gamma \| w \|_t + \gamma \tag{7.3.41}$$

Using lemma 3.6.4 and the fact that $\hat{L}_c^{-1}$ is strictly proper and stable, we get

$$| \Phi^T \hat{L}^{-1}(w) - \hat{L}^{-1}(\Phi^T w)| \le \beta \| w \|_t + \beta \tag{7.3.42}$$

(b) Regularity of $w, \Phi^T w$

The differential equation for $z_1 = (y, \dot{y}, \ldots, y^{\gamma-1})^T$ is

$$z_1 = \hat{M}(s) \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ s^{\gamma - 1} \end{bmatrix} (\Phi^T w) + \begin{bmatrix} y_m \\ \cdot \\ \cdot \\ \cdot \\ y_m^{\gamma - 1} \end{bmatrix} \tag{7.3.43}$$

Since Φ is bounded and $y_m, \ldots, y_m^{\gamma - 1}$ are bounded by hypothesis, and $s^k \hat{M}(s)$ are all proper stable transfer functions, we have that

$$\| z_1 \|_t \le k \| w \|_t + k \tag{7.3.44}$$

Using (7.3.44) in the exponentially minimum phase zero-dynamics

$$\dot{z}_2 = \Psi(z_1, z_2) \tag{7.3.45}$$

we get

$$\| z_2 \|_t \le k \| w \|_t + k \tag{7.3.46}$$

Equations similar to (7.3.44), (7.3.46) can also be obtained for $\dot{z}_1, \dot{z}_2$ since the transfer functions $\hat{M}, \ldots, s^{\gamma - 1}\hat{M}$ are *strictly* proper. Combining (7.3.44) and (7.3.46), and noting that x is a diffeomorphism of z_1, z_2 we see that

$$\| x \|_t \leq k \| w \|_t + k \tag{7.3.47}$$

and

$$\| \dot{x} \|_t \leq k \| w \|_t + k \tag{7.3.48}$$

Using the hypotheses that $\| \partial w / \partial x \|$ and $\| \partial w / \partial \theta \|$ are bounded and (7.3.48) we get

$$\| \dot{w} \|_t \leq k \| w \|_t + k \tag{7.3.49}$$

Thus w is regular $\Rightarrow \xi = \hat{L}^{-1} w$ is regular by corollary 3.6.3. For $\Phi^T w$, note that

$$\frac{d}{dt}(\Phi^T w) = \dot{\Phi}^T w + \Phi^T \dot{w} \tag{7.3.50}$$

Using (7.3.49), and $\Phi, \dot{\Phi} \in L_\infty$ we get

$$\| \frac{d}{dt} \Phi^T w \|_t \leq k \| w \|_t + k \tag{7.3.51}$$

But from (7.3.43) and (7.3.45) we get that

$$\| x \|_t \leq k \| \Phi^T w \|_t + k \tag{7.3.52}$$

so that

$$\| w \|_t \leq k \| \Phi^T w \|_t + k \tag{7.3.53}$$

Combining (7.3.53) with (7.3.51) yields the regularity of $\Phi^T w$.

(c) Stability Proof

From the regularity of ξ, $\Phi^T w$, one can establish that $\Phi^T \xi / 1 + \| \xi \|_t$ has bounded derivative and so is uniformly continuous. By theorem 2.4.6,

$$| e_1(t) | = | \Phi^T \xi(t) | \leq \beta (1 + \| \xi \|_t) \tag{7.3.54}$$

where $\beta \to 0$ as $t \to \infty$.

Now

$$e_0 = e_1 + \Phi^T \hat{L}^{-1}(w) - \hat{L}^{-1}(\Phi^T w) \tag{7.3.55}$$

Using (7.3.42),

$$| e_0 | \leq | e_1 | + \beta \| w \|_t + \beta$$

Using (7.3.53), we have

$$| e_0 | \leq | e_1 | + \beta \| \Phi^T w \|_t + \beta \tag{7.3.56}$$

Applying the BOBI lemma (lemma 3.6.2) to

$$e_0 = \hat{L}^{-1}(s)(\Phi^T w)$$

along with the established regularity of $\Phi^T w$, we get

$$\| \Phi^T w \|_t \leq k \| e_0 \|_t + k \tag{7.3.57}$$

Using (7.3.57) in (7.3.56)

$$| e_0 | \leq | e_1 | + \beta \| e_0 \|_t + \beta \tag{7.3.58}$$

and using (7.3.54) for e_1, we find

$$| e_0 | \leq \beta \| e_0 \|_t + \beta + \beta \| \xi \|_t \tag{7.3.59}$$

Since ξ is related to w by stable filtering

$$\| \xi \|_t \leq k \| w \|_t + k \tag{7.3.60}$$

Using the estimate (7.3.53), followed by (7.3.57) in (7.3.59) yields

$$| e_0 | \leq \beta \| e_0 \|_t + \beta \tag{7.3.61}$$

Since $\beta \to 0$ as $t \to \infty$, we see from (7.3.61) that e_0 goes to zero as $t \to \infty$ (as in proof of theorem 3.7.1, using lemma 3.6.6). This in turn can be easily verified to yield bounded w, x. □

Comments

a) The parameter update law (7.3.35) appears not to take into account *prior parameter* information such as the initial existence of θ_i^*, θ_j^*, $\theta_i^* \theta_j^*$ and so on. It is important, however, to note that the best estimate of $\theta_i^* \theta_j^*$ in the transient period may not be $\theta_i(t)\, \theta_j(t)$. Since parameter convergence is not guaranteed in the proof of theorem 7.3.3, it may also not be a good idea to constrain the estimate of $\theta_i^* \theta_j^*$ to be close to $\theta_i \theta_j$. Note however, that the number of parameters increases very rapidly with γ.

b) In several problems, it turns out that $L_f^\gamma h$ and $L_g L_f^{\gamma-1} h$ depend linearly on some unknown parameters. It is then clear that the development of the previous theorem can be carried through.

c) Thus far, we have only assumed parameter uncertainty in f and g, but not in h. It is not hard to see that if h depends linearly on unknown parameters, then we can mimic the aforementioned procedure quite easily.

d) Parameter convergence can be guaranteed in theorem 7.3.3 above if w is persistently exciting in the usual sense (cf (7.3.14)).

7.3.3. Adaptive Control of MIMO Systems Decouplable by Static State Feedback

From the preceding discussion, it is easy to see how the linearizing, decoupling static state feedback control law for minimum, phase, square systems can be made adaptive—by replacing the control law of (7.2.12) by

$$u = (A(x))_e^{-1}\left[- \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \cdot \\ \cdot \\ \cdot \\ L_f^{\gamma_p} h_p \end{bmatrix}_e + v \right] \tag{7.3.62}$$

Recall that if $A(x)$ is invertible, then the linearizing control law is also the decoupling control law. Thus, if $A(x)$ and the $L_f^{\gamma_i} h_i$'s depend linearly on certain unknown parameters, the schemes of the previous sections (those of Section 7.3.1 if $v_1 = v_2 = \cdots = v_p = 1$, and those of Section 7.3.2 in other cases) can be readily adapted. The details are more notationally cumbersome than insightful. Therefore, we choose not to discuss them here. Instead, we will illustrate our theory on an important class of such systems which partially motivated the present work (see Craig, Hsu, and Sastry [1987])—the adaptive control of rigid link robot manipulators. We sketch only a few of the details of the application relevant to our present context, the interested reader is referred to the paper referenced previously.

If $q \in \mathbb{R}^n$ represents the joint angles of a rigid link robot manipulator, its dynamics may be described by an equation of the form

$$M(q)\ddot{q} + C(q, \dot{q}) = u \tag{7.3.63}$$

In (7.3.63), $M(q) \in \mathbb{R}^{n\times n}$ is the positive definite inertia matrix, $C(q, \dot{q})$ represent the Coriolis, gravity and friction terms, and $u \in \mathbb{R}^n$ represents the control input to the joint motors (torques). In applications, $M(q)$ and $C(q, \dot{q})$ are not known exactly, but fortunately they depend linearly on unknown parameters such as payloads, frictional coefficients, ..., so that

$$M(q) = \sum_{i=1}^{n_1} \theta_i^{(2)^*} M_i(q) \tag{7.3.64}$$

$$C(q, \dot{q}) = \sum_{j=1}^{n_2} \theta_j^{(1)^*} C_j(q, \dot{q}) \tag{7.3.65}$$

Writing the equation (7.3.63) in state space form with $x^T = (q^T, \dot{q}^T)$ and $y = q$, we see that the system is decouplable in the sense of Section 7.2 with $\gamma_1 = \cdots = \gamma_n = 2$, and

$$A(x) = M^{-1}(q) \tag{7.3.66}$$

$$\begin{bmatrix} L_f^{\gamma_1} h_1 \\ \cdot \\ \cdot \\ \cdot \\ L_f^{\gamma_n} h_n \end{bmatrix} = -M^{-1}(q)C(q, \dot{q}) \tag{7.3.67}$$

while the decoupling control law is given by

$$u = C(q, \dot{q}) + M(q)v \tag{7.3.68}$$

Note that the quantities in equation (7.3.66) depend on a complicated fashion on the unknown parameters $\theta^{(1)^*}$, $\theta^{(2)^*}$ while the equation (7.3.68) depends on them *linearly*. For the sake of tracking, v is chosen to be

$$v = \ddot{q}_m + \alpha_2(\dot{q}_m - \dot{q}) + \alpha_1(q_m - q) \tag{7.3.69}$$

and the overall control law (7.3.68), (7.3.69) is referred to as the *computed torque* scheme.

To make the scheme adaptive, the law (7.3.68) is replaced by

$$u = C_e(q, \dot{q}) + M_e(q)v \tag{7.3.70}$$

Let $e_0 = q_m - q$ so that

$$\ddot{e}_0 + \alpha_2 \dot{e}_0 + \alpha_1 e_0 = M_e^{-1}(q) \sum_{j=1}^{n_1} C_j(q, \dot{q})\, \phi_j^{(1)} + M_e^{-1}(q) \sum_{i=1}^{n_2} M_i(q)\ddot{q}\, \phi_i^{(2)} \tag{7.3.71}$$

This may be abbreviated as

$$\ddot{e}_0 + \alpha_2 \dot{e}_0 + \alpha_1 e_0 = W\Phi \tag{7.3.72}$$

where $W \in \mathbb{R}^{n\times(n_1+n_2)}$ is a function of q, $\dot{q}$, and $\ddot{q}$, and Φ is the parameter error vector. The parameter update law

$$\dot{\Phi} = -W^T e_1 \tag{7.3.73}$$

where $e_1 = \dot{e}_0 + \beta_1 e_0$ is chosen so that $(s+\beta_1)/(s^2+\alpha_2 s+\alpha_1)$ is strictly positive real. This can be shown to yield bounded tracking. *The error augmentation of Section 7.3.2 is not necessary in this application* since both $y, \dot{y}$ are available as states so that the $L_f h_i$'s do not have to be estimated. Note that the system is minimum phase—there are in fact no zero dynamics at all. It is, however, unfortunate that the signal W is a function of $\ddot{q}$—but this is caused by the form of the equations and may be avoided by modifying the scheme as in the input error approach (cf. Hsu *et al* [1987]). As in other examples, it is important to keep $M_e(q)$ from becoming singular, using prior parameter bounds.

7.4 CONCLUSIONS

We have presented some initial results on the use of parameter adaptive control for obtaining asymptotically exact cancellation in linearizing control laws. We considered the class of continuous time systems decouplable by static state feedback. The extension to continuous time systems not decouplable by static state feedback is not as obvious for two reasons

a) The different matrices involved in the development of the control laws in this case, namely, $T(x)$, $C(x, w_1)$, $B(x, w_1)$ depend in extremely complicated fashion on the unknown parameters.

b) While the "true" $A(x)$ may have rank less than p, its estimate $A_e(x)$ during the course of adaptation may well be full rank, in which case the procedure of Section 7.2.1 cannot be followed.

The discrete time and sampled data case are also not obvious for similar reasons:

a) The non-adaptive theory, as discussed in Monaco, Normand-Cyrot & Stornelli [1986] is fairly complicated since

$$y_{k+1} = h \circ (f(x_k) + g(x_k)u_k) \tag{7.4.1}$$

is not linear in u_k in the discrete time case and a formal series for (7.4.1) in u_k needs to be obtained (and inverted!) for the linearization. Consequently the parametric dependence of the control law is complex.

b) The notions of zero-dynamics are not as yet completely developed. Further, even in the linear case, the zeros of a sampled system can be outside the unit disc even when the continuous time system is minimum phase and the sampling is fast enough (Astrom, Hagander & Sternby [1984]).

Thus, the present chapter is only a first step in the development of a comprehensive theory of adaptive control for linearizable systems.