2.8 CONCLUSIONS

In this chapter, we derived a simple identification scheme for SISO LTI plants. The scheme involved a generic linear error equation, relating the identifier error, the regressor and the parameter error. Several gradient and least-squares algorithms were reviewed and common properties were established, that are valid under general conditions. It was shown that for any of these algorithms and provided that the regressor was a bounded function of time, the identifier error converged to zero as t approached infinity. The parameter error was also guaranteed to remain bounded. When the regressor was not bounded, but satisfied a regularity condition, then it was shown that a normalized error still converged to zero.

The exponential convergence of the parameter error to its nominal value followed from a persistency of excitation condition on the regressor. Guaranteed rates of exponential convergence were also obtained and showed the influence of various design parameters. In particular, the reference input was found to be a dominant factor influencing the parameter convergence.

The stability and convergence properties were further extended to strictly positive real error equations. Although more complex to analyze, the SPR error equation was found to have similar stability and convergence properties. In particular, PE appeared as a fundamental condition to guarantee exponential parameter convergence.

Finally, the PE conditions were transformed into conditions on the input. We assumed stationarity of the input, so that a frequency-domain analysis could be carried out. It was shown that parameter convergence was guaranteed, if the input contained the same number of spectral components as there were unknown parameters. If the input was a sum of sinusoids, for example, their number should be greater than or equal to the order of the plant.

CHAPTER 3
ADAPTIVE CONTROL

3.0 INTRODUCTION

In this chapter, we derive and analyze algorithms for adaptive control. Our attention is focused on model reference adaptive control. Then, the objective is to design an adaptive controller such that the behavior of the controlled plant remains close to the behavior of a desirable model, despite uncertainties or variations in the plant parameters. More formally, a *reference model* $\hat{M}$ is given, with input $r(t)$ and output $y_m(t)$. The unknown *plant* $\hat{P}$ has input $u(t)$ and output $y_p(t)$. The control objective is to design $u(t)$ such that $y_p(t)$ asymptotically tracks $y_m(t)$, with all generated signals remaining bounded.

We will consider linear time invariant systems of arbitrary order, and establish the stability and convergence properties of the adaptive algorithms. In this section however, we start with an informal discussion for a first order system with two unknown parameters. This will allow us to introduce the algorithms and the stability results in a simpler context.

Consider a first order single-input single-output (SISO) linear time invariant (LTI) plant with transfer function

$$\hat{P} = \frac{k_p}{s + a_p} \tag{3.0.1}$$

where k_p and a_p are unknown. The reference model is a stable SISO LTI system of identical order

$$\hat{M} = \frac{k_m}{s + a_m} \tag{3.0.2}$$

where k_m and $a_m > 0$ are arbitrarily chosen by the designer. In the time domain, the plant is described by

$$\dot{y}_p(t) = -a_p\, y_p(t) + k_p\, u(t) \tag{3.0.3}$$

and the reference model by

$$\dot{y}_m(t) = -a_m\, y_m(t) + k_m\, r(t) \tag{3.0.4}$$

The next steps are similar to those followed for model reference identification in Section 2.0. Let the control input be given by

$$u(t) = c_0(t)\, r(t) + d_0(t)\, y_p(t) \tag{3.0.5}$$

the motivation being that there exist *nominal* parameter values

$$c_0^* = \frac{k_m}{k_p} \qquad d_0^* = \frac{a_p - a_m}{k_p} \tag{3.0.6}$$

such that the closed-loop transfer function matches the reference model transfer function. Specifically, (3.0.3) and (3.0.5) yield

$$\begin{aligned}\dot{y}_p(t) &= -a_p\, y_p(t) + k_p\, (\, c_0(t)\, r(t) + d_0(t)\, y_p(t)\,) \\ &= -(\, a_p - k_p\, d_0(t)\,)\, y_p(t) + k_p\, c_0(t)\, r(t)\end{aligned} \tag{3.0.7}$$

which becomes

$$\dot{y}_p(t) = -a_m\, y_p(t) + k_m\, r(t) \tag{3.0.8}$$

when $c_0(t) = c_0^*$, $d_0(t) = d_0^*$.

For the analysis, it is convenient to introduce an error formulation. Define the *output error*

$$e_0 = y_p - y_m \tag{3.0.9}$$

and the *parameter error*

$$\phi = \begin{bmatrix} \phi_r(t) \\ \phi_y(t) \end{bmatrix} = \begin{bmatrix} c_0(t) - c_0^* \\ d_0(t) - d_0^* \end{bmatrix} \tag{3.0.10}$$

Subtracting (3.0.4) from (3.0.7)

$$\dot{e}_0 = -a_m\, (y_p - y_m) + (a_m - a_p + k_p\, d_0)\, y_p + k_p\, c_0\, r - k_m\, r$$

$$\begin{aligned} &= -a_m\, e_0 + k_p \left[(c_0 - c_0^*)\, r + (d_0 - d_0^*)\, y_p \right] \\ &= -a_m\, e_0 + k_p(\phi_r r + \phi_y\, y_p)\end{aligned} \tag{3.0.11}$$

We may represent (3.0.11) in compact form as

$$\begin{aligned} e_0 &= \frac{k_p}{s + a_m} \hat{M}\, (\phi_r r + \phi_y\, y_p) \\ &= \frac{k_p}{k_m} \hat{M}\, (\phi_r r + \phi_y\, y_p) = \frac{1}{c_0^*} \hat{M}\, (\phi_r r + \phi_y\, y_p)\end{aligned} \tag{3.0.12}$$

Although this notation will be very convenient in this book, we caution the reader that it mixes time domain operations $(\phi_r r + \phi_y\, y_p)$ and filtering by the LTI operator $\hat{M}$.

Equation (3.0.12) is of the form of the *SPR error equation* of Chapter 2. Therefore, we tentatively choose the update laws

$$\begin{aligned}\dot{c}_0 &= -g\, e_0\, r \\ \dot{d}_0 &= -g\, e_0\, y_p \qquad g > 0\end{aligned} \tag{3.0.13}$$

assuming that $k_p / k_m > 0$ and that $\hat{M}$ is SPR. The first condition requires prior knowledge about the plant (sign of the *high frequency gain* k_p), while the second condition restricts the class of models which may be chosen.

Note that there is a significant difference with the model reference identification case, namely that the signal y_p which appears in (3.0.12) is not exogeneous, but is itself a function of e_0. However, the stability proof proceeds along exactly the same lines. First assume that r is bounded, so that y_m is also bounded. The adaptive system is described by (3.0.3)–(3.0.5) and (3.0.13). Alternatively, the error formulation is

$$\begin{aligned}\dot{e}_0 &= -a_m\, e_0 + k_p(\phi_r r + \phi_y\, e_0 + \phi_y\, y_m) \\ \dot{\phi}_r &= -g\, e_0\, r \\ \dot{\phi}_y &= -g\, e_0^2 - g\, e_0\, y_m\end{aligned} \tag{3.0.14}$$

In this representation, the right-hand sides only contain states (e_0, ϕ_r, ϕ_y) and exogeneous signals (r, y_m). Consider then the Lyapunov function

$$v(e_0, \phi_r, \phi_y) = \frac{e_0^2}{2} + \frac{k_p}{2g}(\phi_r^2 + \phi_y^2) \tag{3.0.15}$$

so that, along the trajections of (3.0.14)

$$\dot{v} = -a_m e_0^2 + k_p \phi_r e_0 r + k_p \phi_y e_0^2 + k_p \phi_y e_0 y_m - k_p \phi_r e_0 r - k_p \phi_y e_0^2 - k_p \phi_y e_0 y_m = -a_m e_0^2 \le 0 \tag{3.0.16}$$

It follows that the adaptive system is stable (in the sense of Lyapunov) and that, for all initial conditions, e_0, ϕ_r and ϕ_y are bounded. From (3.0.14), $\dot{e}_0$ is also bounded. Since v is monotonically decreasing and bounded below, $\lim v(t)$ as $t \to \infty$ exists, so that $e_0 \in L_2$. Since $e_0 \in L_2 \cap L_\infty$, and $\dot{e}_0 \in L_\infty$, it follows that $e_0 \to 0$ as $t \to \infty$.

The above approach is elegant, and relatively simple. However, it is not straightforward to extend it to the case when the relative degree of the plant is greater than 1. Further, it places restrictions on the reference model which are in fact not necessary. We now discuss an alternate approach which we will call the *input error approach*. The resulting scheme is slightly more complex for this first order example, but has significant advantages, which will be discussed later.

Instead of using e_0 in the adaptation procedure, we use the error

$$e_2 = c_0 y_p + d_0 \hat{M}(y_p) - \hat{M}(u) \tag{3.0.17}$$

The motivation behind this choice will be made clear from the analysis. Equation (3.0.17) determines how e_2 is calculated in the implementation (as (3.0.9) in the output error scheme). For the derivation of the adaptation scheme and for the analysis, we relate e_2 to the parameter error (as (3.0.12) in the output error scheme). First note that

$$\begin{aligned} \hat{M} &= \frac{k_m}{s+a_m} = \frac{k_m}{k_p}\frac{s+a_p}{s+a_m}\frac{k_p}{s+a_p} \\ &= \frac{k_m}{k_p}\frac{k_p}{s+a_p} + \frac{k_m}{k_p}\frac{a_p-a_m}{s+a_m}\frac{k_p}{s+a_p} \\ &= \frac{k_m}{k_p}\hat{P} + \frac{a_p-a_m}{k_p}\hat{M}\hat{P} \\ &= c_0^* \hat{P} + d_0^* \hat{M}\hat{P} \end{aligned} \tag{3.0.18}$$

where we used (3.0.6). Therefore, applying (3.0.18) to the signal u

$$\hat{M}(u) = c_0^* y_p + d_0^* \hat{M}(y_p) \tag{3.0.19}$$

and with (3.0.17)

$$\begin{aligned} e_2 &= (c_0 - c_0^*) y_p + (d_0 - d_0^*)\hat{M}(y_p) \\ &= \phi_r y_p + \phi_y \hat{M}(y_p) \end{aligned} \tag{3.0.20}$$

Equation (3.0.20) is of the form of the *linear error equation* studied in Chapter 2. Therefore, we may now use any of the identification algorithms, including the least-squares algorithms. No condition is posed on the reference model by the identification/adaptation algorithm. Proving stability is however more difficult, and is not addressed in this introduction.

The algorithms described above are both *direct algorithms*, for which update laws are designed to directly update the controller parameters c_0 and d_0. An alternate approach is the *indirect approach*. Using any of the procedures discussed in Chapter 2, we may design a recursive identifier to provide estimates of the plant parameters k_p and a_p. If k_p and a_p were known, the equations in (3.0.6) would determine the nominal controller parameters c_0^* and d_0^*. In an indirect approach, one replaces k_p and a_p in (3.0.6) by their estimates, thereby defining c_0 and d_0. This is a very intuitive approach to adaptive control which we will also discuss in this chapter.

After extending the above algorithms to more general schemes for plants of arbitrary order, we will study the stability and convergence properties of the adaptive systems. Unfortunately, the simple Lyapunov stability proof presented for the output error scheme does not extend to the general case, or to the other schemes. Instead, we will use tools from functional analysis, together with a set of standard lemmas which we will first derive. The global stability of the adaptive control schemes will be established. Conditions for parameter convergence will also follow, together with input signal conditions similar to those encountered in Chapter 2 for identification.

3.1 MODEL REFERENCE ADAPTIVE CONTROL PROBLEM

We now turn to the general model reference adaptive control problem considered in this chapter. The following assumptions will be in effect.

Assumptions

(A1) **Plant Assumptions**

The plant is a single-input, single-output (SISO), linear time-invariant (LTI) system, described by a transfer function

$$\frac{\hat{y}_p(s)}{\hat{u}(s)} = \hat{P}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} \tag{3.1.1}$$

where $\hat{n}_p(s)$, $\hat{d}_p(s)$ are monic, coprime polynomials of degree m and n, respectively. The plant is strictly proper and minimum phase. The sign of the so-called *high-frequency gain* k_p is known and, without loss of generality, we will assume $k_p > 0$.

(A2) Reference Model Assumptions

The reference model is described by

$$\frac{\hat{y}_m(s)}{\hat{r}(s)} = \hat{M}(s) = k_m \frac{\hat{n}_m(s)}{\hat{d}_m(s)} \tag{3.1.2}$$

where $\hat{n}_m(s)$, $\hat{d}_m(s)$ are monic, coprime polynomials of degree m and n respectively (that is, the same degrees as the corresponding plant polynomials). The reference model is stable, minimum phase and $k_m > 0$.

(A3) Reference Input Assumptions

The reference input $r(.)$ is piecewise continuous and bounded on $\mathbb{R}_+$.

Note that $\hat{P}(s)$ is assumed to be minimum phase, but is *not* assumed to be stable.

3.2 CONTROLLER STRUCTURE

To achieve the control objective, we consider the controller structure shown in Figure 3.1.

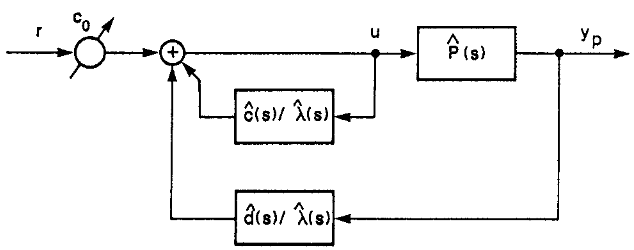

Figure 3.1: Controller Structure

By inspection of the figure, we see that

$$u = c_0 r + \frac{\hat{c}(s)}{\hat{\lambda}(s)}(u) + \frac{\hat{d}(s)}{\hat{\lambda}(s)}(y_p) \tag{3.2.1}$$

where c_0 is a scalar, $\hat{c}(s)$, $\hat{d}(s)$ and $\hat{\lambda}(s)$ are polynomials of degrees $n-2$, $n-1$ and $n-1$, respectively. From (3.2.1),

$$u = \frac{\hat{\lambda}}{\hat{\lambda}-\hat{c}}\left(c_0 r + \frac{\hat{d}}{\hat{\lambda}}(y_p)\right) \tag{3.2.2}$$

which is shown in Figure 3.2.

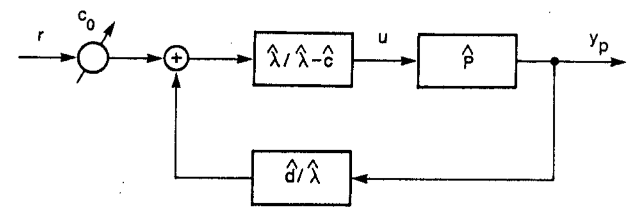

Figure 3.2: Controller Structure—Equivalent Form

Since

$$y_p = k_p \frac{\hat{n}_p}{\hat{d}_p}(u) \tag{3.2.3}$$

the transfer function from r to y_p is

$$\frac{\hat{y}_p}{\hat{r}} = \frac{c_0 k_p \hat{\lambda} \hat{n}_p}{(\hat{\lambda}-\hat{c})\hat{d}_p - k_p \hat{n}_p \hat{d}} \tag{3.2.4}$$

Note that the derivation of (3.2.4) relies on the cancellation of polynomials $\hat{\lambda}(s)$. Physically, this would correspond to the exact cancellation of modes of $\hat{c}(s)/\hat{\lambda}(s)$ and $\hat{d}(s)/\hat{\lambda}(s)$. For numerical considerations, we will therefore require that $\hat{\lambda}(s)$ is a Hurwitz polynomial.

The following proposition indicates that the controller structure is adequate to achieve the control objective, that is, that it is possible to make the transfer function from r to y_p equal to $\hat{M}(s)$. For this, it is clear from (3.2.4) that $\hat{\lambda}(s)$ must contain the zeros of $\hat{n}_m(s)$, so that we write

$$\hat{\lambda}(s) = \hat{\lambda}_0(s)\hat{n}_m(s) \tag{3.2.5}$$

where $\hat{\lambda}_0(s)$ is an arbitrary Hurwitz polynomial of degree $n-m-1$.

Proposition 3.2.1 Matching Equality

There exist unique c_0^*, $\hat{c}^*(s)$, $\hat{d}^*(s)$ such that the transfer function from $r \to y_p$ is $\hat{M}(s)$.

Proof of Proposition 3.2.1

1. Existence

The transfer function from r to y_p is $\hat{M}$ if and only if the following *matching equality* is satisfied

$$(\hat{\lambda} - \hat{c}^*)\hat{d}_p - k_p \hat{n}_p \hat{d}^* = c_0^* \frac{k_p}{k_m} \hat{\lambda}_0 \hat{n}_p \hat{d}_m \tag{3.2.6}$$

The solution can be found by inspection. Divide $\hat{\lambda}_0 \hat{d}_m$ by $\hat{d}_p$, let $\hat{q}$ be the quotient (of degree $n - m - 1$) and $-k_p \hat{d}^*$ the remainder (of degree $n - 1$). Thus $\hat{d}^*$ is given by

$$\hat{d}^* = \frac{1}{k_p}(\hat{q}\,\hat{d}_p - \hat{\lambda}_0 \hat{d}_m) \tag{3.2.7}$$

Let $\hat{c}^*$ (of degree $n - 2$), c_0^* be given by

$$\hat{c}^* = \hat{\lambda} - \hat{q}\,\hat{n}_p \tag{3.2.8}$$

$$c_0^* = \frac{k_m}{k_p} \tag{3.2.9}$$

Equations (3.2.7)–(3.2.9) define a solution to (3.2.6), as can easily be seen by substituting c_0^*, c^* and d^* in (3.2.6).

2. Uniqueness

Assume that there exist $c_0 = c_0^* + \delta c_0$, $\hat{c} = \hat{c}^* + \delta\hat{c}$, $\hat{d} = \hat{d}^* + \delta\hat{d}$ satisfying (3.2.6). The following equality must then be satisfied

$$\delta\hat{c}\,\hat{d}_p + k_p \hat{n}_p \delta\hat{d} = -\delta c_0 \frac{k_p}{k_m} \hat{\lambda}_0 \hat{n}_p \hat{d}_m \tag{3.2.10}$$

Recall that $\hat{d}_p$, $\hat{n}_p$, $\hat{\lambda}_0$ and $\hat{d}_m$ have degrees n, m, $n - m - 1$ and n, respectively, with $m \le n - 1$, and $\delta\hat{c}$ and $\delta\hat{d}$ have degrees at most $n - 2$ and $n - 1$. Consequently, the right-hand side is a polynomial of degree $2n - 1$ and the left-hand side is a polynomial of degree at most $2n - 2$. No solution exists unless $\delta c_0 = 0$, so that c_0^* is unique. Let, then, $\delta c_0 = 0$, so that (3.2.10) becomes

$$\frac{\delta\hat{c}}{\delta\hat{d}} = -k_p \frac{\hat{n}_p}{\hat{d}_p} = -\hat{P} \tag{3.2.11}$$

This equation has no solution since $\hat{n}_p$, $\hat{d}_p$ are coprime, so that $\hat{c}^*$ and $\hat{d}^*$ are also unique. □

Comments

a) The coprimeness of $\hat{n}_p$, $\hat{d}_p$ is only necessary to guarantee a *unique* solution. If this assumption is not satisfied, a solution can still be found using (3.2.7)–(3.2.9). Equation (3.2.11) characterizes the set of solutions in this case.

b) Using (3.2.2), the controller structure can be expressed as in Figure 3.2, with a forward block $\hat{\lambda}/\hat{\lambda} - \hat{c}$ and a feedback block $\hat{d}/\hat{\lambda}$. When matching with the model occurs, (3.2.7), (3.2.8) show that the compensator becomes

$$\frac{\hat{\lambda}}{\hat{\lambda} - \hat{c}^*} = \frac{\hat{\lambda}_0 \hat{n}_m}{\hat{q}\,\hat{n}_p} \tag{3.2.12}$$

and

$$\frac{\hat{d}^*}{\hat{\lambda}} = \frac{1}{k_p} \frac{\hat{q}\,\hat{d}_p - \hat{\lambda}_0 \hat{d}_m}{\hat{\lambda}_0 \hat{n}_m} \tag{3.2.13}$$

Thus the forward block actually cancels the zeros of $\hat{P}$ and replaces them by the zeros of $\hat{M}$.

c) The transfer function from r to y_p is of order n, while the plant and controller have $3n - 2$ states. It can be checked (see Section 3.5) that the $2n - 2$ extra modes are those of $\hat{\lambda}$, $\hat{\lambda}_0$ and $\hat{n}_p$. The modes corresponding to $\hat{\lambda}$, $\hat{\lambda}_0$ are stable by choice and those of $\hat{n}_p$ are stable by assumption (A1).

d) The structure of the controller is not unique. In particular, it is equivalent to the familiar structure found, for example, in Callier & Desoer [1982], p. 164, and represented in Figure 3.3. The polynomials found in this case are related to the previous ones through

$$\hat{n}_\pi = c_0 \hat{\lambda} \qquad \hat{d}_c = \hat{\lambda} - \hat{c} \qquad \hat{n}_f = -\hat{d} \tag{3.2.14}$$

The motivation in using the previous controller structure is to obtain an expression that is *linear* in the unknown parameters. These parameters are the coefficients of the polynomials $\hat{c}$, $\hat{d}$ and the gain c_0. The expression in (3.2.1) shows that the control signal is the sum of the parameters multiplied by known or reconstructible signals.

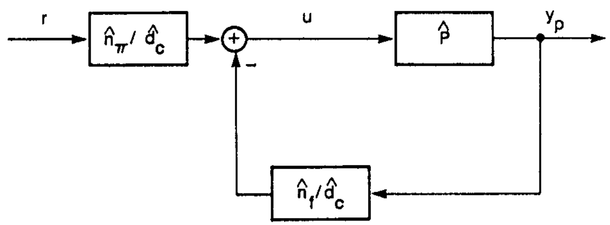

Figure 3.3: Alternate Controller Structure

State-Space Representation

To make this more precise, we consider a state-space representation of the controller. Choose $\Lambda \in \mathbb{R}^{n-1 \times n-1}$ and $b_\lambda \in \mathbb{R}^{n-1}$, such that (Λ, b_λ) is in controllable canonical form as in (2.2.9) and $\det(sI - \Lambda) = \hat{\lambda}(s)$. It follows that

$$(sI - \Lambda)^{-1} b_\lambda = \frac{1}{\hat{\lambda}(s)} \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ \cdot \\ s^{n-2} \end{bmatrix} \tag{3.2.15}$$

Let $c \in \mathbb{R}^{n-1}$ be the vector of coefficients of the polynomial $\hat{c}(s)$, so that

$$\frac{\hat{c}(s)}{\hat{\lambda}(s)} = c^T (sI - \Lambda)^{-1} b_\lambda \tag{3.2.16}$$

Consequently, this transfer function can be realized by

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\frac{\hat{c}}{\hat{\lambda}}(u) = c^T w^{(1)} \tag{3.2.17}$$

where the state $w^{(1)} \in \mathbb{R}^{n-1}$ and the initial condition $w^{(1)}(0)$ is arbitrary. Similarly, there exist $d_0 \in \mathbb{R}$ and $d \in \mathbb{R}^{n-1}$, such that

$$\frac{\hat{d}(s)}{\hat{\lambda}(s)} = d_0 + d^T (sI - \Lambda)^{-1} b_\lambda \tag{3.2.18}$$

and

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$\frac{\hat{d}}{\hat{\lambda}}(y_p) = d_0 y_p + d^T w^{(2)} \tag{3.2.19}$$

where the state $w^{(2)} \in \mathbb{R}^{n-1}$ and the initial condition $w^{(2)}(0)$ is arbitrary. The controller can be represented as in Figure 3.4, with

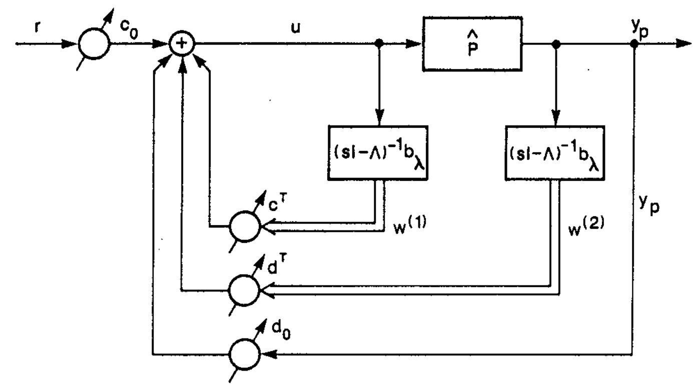

Figure 3.4: Controller Structure—Adaptive Form

$$\begin{aligned} u &= c_0 r + c^T w^{(1)} + d_0 y_p + d^T w^{(2)} \\ &:= \theta^T w \end{aligned} \tag{3.2.20}$$

where

$$\theta^T := (c_0, \bar{\theta}^T) := (c_0, c^T, d_0, d^T) \in \mathbb{R}^{2n} \tag{3.2.21}$$

is the vector of *controller parameters* and

$$w^T := (r, \bar{w}^T) := (r, w^{(1)^T}, y_p, w^{(2)^T}) \in \mathbb{R}^{2n} \tag{3.2.22}$$

is a vector of signals that can be obtained without knowledge of the plant parameters. Note the definitions of $\bar{\theta}$ and $\bar{w}$ which correspond to the vectors θ and w with their first components removed.

In analogy to the previous definitions, we let

$$\theta^{*^T} := (c_0^*, \bar{\theta}^{*^T}) := (c_0^*, c^{*^T}, d_0^*, d^{*^T}) \in \mathbb{R}^{2n} \tag{3.2.23}$$

be the vector of *nominal* controller parameters that achieves a matching of the transfer function $r \rightarrow y_p$ to the model transfer function $\hat{M}$. We also define the *parameter errors*

$$\phi := \theta - \theta^* \in \mathbb{R}^{2n} \qquad \bar{\phi} := \bar{\theta} - \bar{\theta}^* \in \mathbb{R}^{2n-1} \tag{3.2.24}$$

The linear dependence of u on the parameters is clear in (3.2.20). In the sequel, we will consider *adaptive* control algorithms and the parameter θ will be a function of time. Similarly, $\hat{c}(s)$, $\hat{d}(s)$ will be polynomials in s whose coefficients vary with time. Equations (3.2.17) and (3.2.19) give a meaning to (3.2.1) in that case.

3.3 ADAPTIVE CONTROL SCHEMES

In Section 3.2, we showed how a controller can be designed to achieve tracking of the reference output y_m by the plant output y_p, when the plant transfer function is known. We now consider the case when the plant is unknown and the control parameters are updated recursively using an identifier. Several approaches are possible. In an indirect adaptive control scheme, the plant parameters (i.e., k_p and the coefficients of $\hat{n}_p(s)$, $\hat{d}_p(s)$) are identified using a recursive identification scheme, such as those described in Chapter 2. The estimates are then used to compute the control parameters through (3.2.7)–(3.2.9).

In a direct adaptive control scheme, an identification scheme is designed that *directly* identifies the controller parameters c_0, c, d_0 and d. A typical procedure is to derive an identifier error signal which depends linearly on the parameter error ϕ. The output error $e_0(t) = y_p(t) - y_m(t)$ is the basis for output error adaptive control schemes such as those of Narendra & Valavani [1978], Narendra, Lin, & Valavani [1980], and Morse [1980]. An output error direct adaptive control scheme and an indirect adaptive control scheme will be described in Sections 3.3.2 and 3.3.3, but we will first turn to an input error direct adaptive control scheme in Section 3.3.1 (this scheme is discussed in Bodson [1986] and Bodson & Sastry [1987]).

Note that we made the distinction between controller and identifier, even in the case of direct adaptive control. The *controller* is by definition the system that determines the value of the control input, using some controller parameters as in a nonadaptive context. The *identifier* obtains estimates of these parameters—directly or indirectly.

As in Chapter 2, we also make the distinction, within the identifier, between the *identifier structure* and the *identification algorithm*. The identifier structure constructs signals which are related by some error equation and are to be used by the identification algorithm. The identification algorithm defines the evolution of the identifier parameters, from which the controller parameters depend. Given an identifier structure with linear error equation for example, several identification algorithms exist from which we can choose (cf. Section 2.3).

Although we make the distinction between controller and identifier, we will see that, for efficiency, some internal signals will be shared by

both systems.

3.3.1 Input Error Direct Adaptive Control

Define

$$r_p = \hat{M}^{-1}(y_p) = \hat{M}^{-1}\hat{P}(u) \tag{3.3.1}$$

and let the *input error* e_i be defined by

$$\begin{aligned} e_i &:= r_p - r \\ &= \hat{M}^{-1}(y_p - y_m) = \hat{M}^{-1}(e_0) \end{aligned} \tag{3.3.2}$$

where $e_0 = y_p - y_m$ is the *output error*.

By definition, an input error adaptive control scheme is a scheme based on this error, or a modification of it.

Input Error and Linear Error Equation

The interest of the input error in (3.3.2) is to lead to a linear error equation such as studied in Section 2.3 in the context of identification. We first present an intuitive derivation of this linear error equation.

Consider the matching equality (3.2.6), and divide both sides by $\hat{\lambda}\hat{d}_p$. Using (3.2.5)

$$1 - \frac{\hat{c}^*}{\hat{\lambda}} - k_p\frac{\hat{n}_p}{\hat{d}_p}\frac{\hat{d}^*}{\hat{\lambda}} = c_0^*\frac{k_p\hat{n}_p}{\hat{d}_p}\frac{\hat{d}_m}{k_m\hat{n}_m} \tag{3.3.3}$$

With the definitions of $\hat{P}$ and $\hat{M}$, (3.3.3) becomes

$$1 - \frac{\hat{c}^*}{\hat{\lambda}} - \frac{\hat{d}^*}{\hat{\lambda}}\hat{P} = c_0^*\hat{M}^{-1}\hat{P} \tag{3.3.4}$$

We may interpret this last equality as an equality of two polynomial ratios, but also as an equality of two LTI system transfer functions. Applying both transfer functions to the signal u, we have

$$u - \frac{\hat{c}^*}{\hat{\lambda}}(u) - \frac{\hat{d}^*}{\hat{\lambda}}(y_p) = c_0^*\hat{M}^{-1}(y_p) \tag{3.3.5}$$

Now, we recall that $\overline{w} \in \mathbb{R}^{2n} - 1$ is given by

$$\overline{w} = \begin{bmatrix} (sI - \Lambda)^{-1}b_\lambda(u) \\ y_p \\ (sI - \Lambda)^{-1}b_\lambda(y_p) \end{bmatrix} \tag{3.3.6}$$

so that, with (3.3.5)

$$\bar{\theta}^{*T}\bar{w} = \frac{\hat{c}^*}{\hat{\lambda}}(u) + \frac{\hat{d}^*}{\hat{\lambda}}(y_p)$$

$$= u - c_0^* \hat{M}^{-1}(y_p) \tag{3.3.7}$$

The control input is given by (3.2.20)

$$u = \theta^T w = c_0 r + \bar{\theta}^T \bar{w} \tag{3.3.8}$$

so that

$$\bar{\theta}^{*T}\bar{w} = c_0 r + \bar{\theta}^T \bar{w} - c_0^* \hat{M}^{-1}(y_p)$$

$$= c_0(r - \hat{M}^{-1}(y_p)) + \bar{\theta}^T \bar{w} + (c_0 - c_0^*)\hat{M}^{-1}(y_p)$$

$$= -c_0 e_i + \bar{\theta}^T \bar{w} + (c_0 - c_0^*)\hat{M}^{-1}(y_p) \tag{3.3.9}$$

We now define the signal

$$z^T := (r_p, \bar{w}^T) := (\hat{M}^{-1}(y_p), \bar{w}^T) \in \mathbb{R}^{2n} \tag{3.3.10}$$

so that (3.3.9) becomes

$$e_i = \frac{1}{c_0}\phi^T z \tag{3.3.11}$$

This equation is of the form of the linear error equation studied in Section 2.3 (the gain $1/c_0$ being known may be merged either with e_i or z, as will be seen later). It could thus be used to derive an identification procedure to identify the parameter θ directly. As presented, however, the scheme and its derivation show several problems:

a) Since the relative degree of $\hat{M}$ is at least 1, its inverse is not proper. Although $\hat{M}^{-1}(.)$ is well defined, provided that the argument is sufficiently smooth, the gain of the operator $\hat{M}^{-1}$ is arbitrarily large at high frequencies. Therefore, due to the presence of measurement noise, the use of $\hat{M}^{-1}$ is not desirable in practice. Although we will use $\hat{M}^{-1}(.)$ in the analysis, we will consider it not implementable, so that r_p and e_i are not available.

b) The derivation of the error equation (3.3.11) relies on (3.3.8) being satisfied at all times. Although this is not a crucial problem, we will discuss the advantages of avoiding it.

c) We were somewhat careless with initial conditions, going from (3.3.4) to (3.3.5), since $\hat{P}$ may be unstable.

We now derive a modified input error, leading to a scheme that does not have the above disadvantages **a)** and **b)**, and resolve the technical question in **c)**.

Fundamental Identity

Since $\hat{M}$ is minimum phase with relative degree $n-m$, for any stable, minimum phase transfer function $\hat{L}^{-1}$ of relative degree $n-m$, the transfer function $\hat{M}\hat{L}$ has a proper and stable inverse. For example, we can let $\hat{L}$ be a Hurwitz polynomial of degree $n-m$. The signal $\hat{L}^{-1}(r_p)$ is available since

$$\hat{L}^{-1}(r_p) = (\hat{M}\hat{L})^{-1}(y_p) \tag{3.3.12}$$

where $(\hat{M}\hat{L})^{-1}$ is a proper, stable transfer function.

Divide both sides of (3.2.6) by $\hat{\lambda}\hat{d}_p\hat{L}$ so that it becomes, using (3.2.5) and the definitions of $\hat{P}$ and $\hat{M}$,

$$\left[\hat{L}^{-1}\frac{\hat{d}^*}{\hat{\lambda}} + c_0^*(\hat{M}\hat{L})^{-1}\right]\hat{P} = \hat{L}^{-1} - \hat{L}^{-1}\frac{\hat{c}^*}{\hat{\lambda}} \tag{3.3.13}$$

Consider (3.3.13) as an equality of two transfer functions. The right-hand side is a stable transfer function, while the left-hand side is possibly unstable (since $\hat{P}$ is not assumed to be stable).

To transform (3.3.13) into an equality in the time domain, care must be taken of the effect of the initial conditions related to the unstable modes of $\hat{P}$. These will be unobservable or uncontrollable, depending on the realization of the transfer function. If the left-hand side is realized by $\hat{P}$ followed by

$$L^{-1}\frac{\hat{d}^*}{\hat{\lambda}} + c_0^*(\hat{M}\hat{L})^{-1}$$

the unstable modes of $\hat{P}$ will be controllable and, therefore, unobservable.

The operator equality (3.3.13) can then be transformed to a signal equality by applying both operators to u, so that

$$\hat{L}^{-1}\frac{\hat{d}^*}{\hat{\lambda}}(y_p) + c_0^*(\hat{M}\hat{L})^{-1}(y_p) = \hat{L}^{-1}(u) - \hat{L}^{-1}\frac{\hat{c}^*}{\hat{\lambda}}(u) + \epsilon(t) \tag{3.3.14}$$

where $\epsilon(t)$ reminds us of the presence of exponentially *decaying* terms due to initial conditions. These are decaying because the transfer functions are stable and the unstable modes are unobservable. Therefore, (3.3.14) is valid for arbitrary initial conditions in the realizations of $\hat{L}^{-1}$, $\hat{\lambda}$, and $(\hat{M}\hat{L})^{-1}$.

Since $\bar{\theta}^*$ is constant, $\bar{\theta}^{*T}\hat{L}^{-1}(\bar{w})$ is given by

$$\bar{\theta}^{*T}\hat{L}^{-1}(\bar{w}) = \hat{L}^{-1}(\bar{\theta}^{*T}\bar{w})$$

$$= \hat{L}^{-1}\left[\frac{\hat{c}^*}{\hat{\lambda}}(u) + \frac{\hat{d}^*}{\hat{\lambda}}(y_p)\right]$$

$$= \hat{L}^{-1}(u) - c_0^*(\hat{M}\hat{L})^{-1}(y_p) + \epsilon(t) \qquad (3.3.15)$$

where we used (3.3.14). Define now

$$v^T := \left[\hat{L}^{-1}(r_p), \hat{L}^{-1}(\bar{w}^T)\right] = \left[(\hat{M}\hat{L})^{-1}(y_p), \hat{L}^{-1}(\bar{w}^T)\right] \in \mathbb{R}^{2n} \qquad (3.3.16)$$

so that (3.3.15) can be written

$$\hat{L}^{-1}(u) = \theta^{*T}v \cancel{+} \epsilon(t) \qquad (3.3.17)$$

where θ^* is defined in (3.2.7)–(3.2.9), with (3.2.23). Equation (3.3.17) is essential to subsequent derivations, so that we summarize the result in the following proposition.

Proposition 3.3.1 Fundamental Identity

Let $\hat{P}$ and $\hat{M}$ satisfy assumptions (A1) and (A2). Let $\hat{L}^{-1}$ be any stable, minimum phase transfer function of relative degree $n - m$. Let v and $\bar{w}$ be as defined by (3.3.16) and (3.3.6), with arbitrary initial conditions in the realizations of the transfer functions. Let θ^* be defined by (3.2.7)–(3.2.9), with (3.2.23).

Then for all piecewise continuous $u \in L_{\infty e}$, (3.3.17) is satisfied.

Input Error Identifier Structure

Equation (3.3.17) is of the form studied in Section 2.3 for recursive identification. Both the signal $\hat{L}^{-1}(u)$ and v are available from measurements, and the expression is linear in the unknown parameter θ^*.

Therefore, we define the *modified input error* to be

$$e_2 := \theta^T v - \hat{L}^{-1}(u) \qquad (3.3.18)$$

so that, using (3.3.17)

$$e_2 = \phi^T v + \epsilon(t) \qquad (3.3.19)$$

which is of the form of the *linear error equation* studied in Section 2.3. Although we considered the input error e_i not to be available, because it would require the realization of a nonproper transfer function, the approximate input error e_2, and the signal v are available, given these considerations.

We also observed in Chapter 2 that standard properties of the identification algorithms are not affected by the $\epsilon(t)$ term. For simplicity, we will omit this term in subsequent derivations. We now consider the practical implementation of the algorithm, with the required assumptions.

Assumptions

The algorithm relies on assumptions (A1)–(A3) and the following additional assumption.

(A4) Bound on the High-Frequency Gain

Assume that an upper bound on k_p is known, that is, that $k_p \le k_{\max}$ for some $k_{\max}$.

The structure of the controller and identifier is shown in Figure 3.5, while the complete algorithm is summarized hereafter. The need for assumption (A4), and for the projection of c_0, will be discussed later, in connection with alternate schemes. It will be more obvious from the proof of stability of the algorithm in Section 3.7.

Input Error Direct Adaptive Control Algorithm—Implementation

Assumptions

(A1)–(A4)

Data

$n, m, k_{\max}$

Input

$r(t), y_p(t) \in \mathbb{R}$

Output

$u(t) \in \mathbb{R}$

Internal Signals

$w(t) \in \mathbb{R}^{2n}\ [w^{(1)}(t), w^{(2)}(t) \in \mathbb{R}^{n-1}]$

$\theta(t) \in \mathbb{R}^{2n}\ [c_0(t), d_0(t) \in \mathbb{R}, c(t), d(t) \in \mathbb{R}^{n-1}]$

$v(t) \in \mathbb{R}^{2n}, e_2(t) \in \mathbb{R}$

Initial conditions are arbitrary, except $c_0(0) \ge c_{\min} = k_m/k_{\max} > 0$.

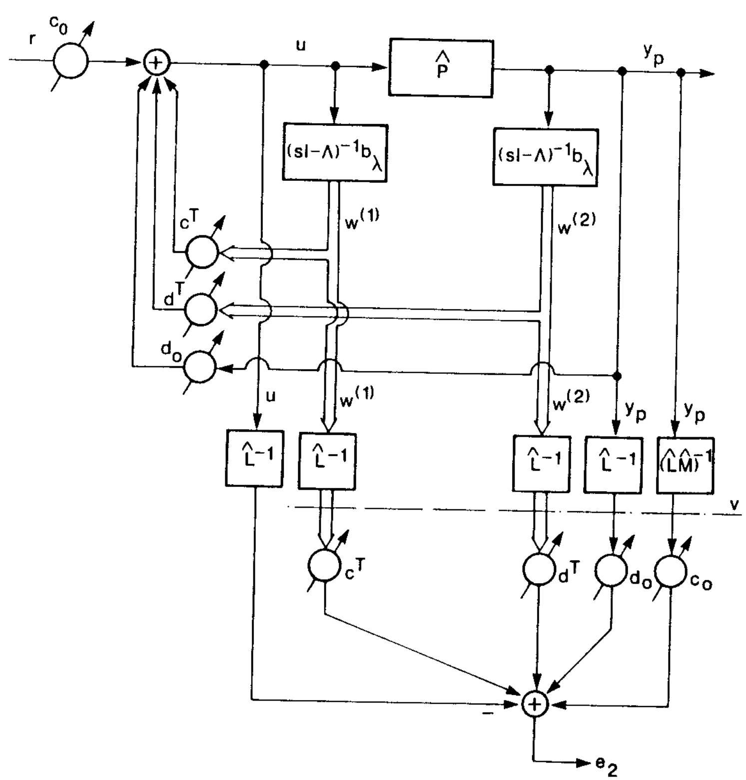

Figure 3.5: Controller and Input Error Identifier Structures

Design Parameters

Choose

- $\hat{M}$ (i.e. k_m, $\hat{n}_m$, $\hat{d}_m$) satisfying (A2).
- $\Lambda \in \mathbb{R}^{n-1 \times n-1}$, $b_\lambda \in \mathbb{R}^{n-1}$ in controllable canonical form, such that det $(sI-\Lambda)$ is Hurwitz and contains the zeros of $\hat{n}_m(s)$.
- $\hat{L}^{-1}$ stable, minimum phase transfer function of relative degree $n-m$.
- $g, \gamma > 0$.

Controller Structure

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$\theta^T = (c_0, c^T, d_0, d^T)$$

$$w^T = (r, w^{(1)}, y_p, w^{(2)})$$

$$u = \theta^T w$$

Identifier Structure

$$v^T = [(\hat{M}\hat{L})^{-1}(y_p), \hat{L}^{-1}(w^{(1)^T}), \hat{L}^{-1}(y_p), \hat{L}^{-1}(w^{(2)^T})]$$

$$e_2 = \theta^T v - \hat{L}^{-1}(u)$$

Normalized Gradient Algorithm with Projection

$$\dot{\theta} = -g \frac{e_2 v}{1 + \gamma v^T v}$$

if $c_0 = c_{\min}$ and $\dot{c}_0 < 0$, then set $\dot{c}_0 = 0$.

□

Comments

Adaptive Observer

The signal generators for $w^{(1)}$ and $w^{(2)}$ ((3.2.17) and (3.2.19)) are almost identical to those used in Chapter 2 for identification of the plant parameters (their dimensions are now $n-1$ instead of n previously). They are shared by the controller and the identifier. The signal generators (sometimes called *state-variable filters) for* $w^{(1)}$ and $w^{(2)}$ form a *generalized observer,* reconstructing the states of the plant in a specific parameterization. This parameterization has the characteristic of allowing the reconstruction of the states without knowledge of the parameters. The states are used for the state feedback of the controller to the input in a certainty equivalence manner, meaning that the parameters used for feedback are the current estimates multiplying the states *as if* they were the true parameters. The identifier with the generalized observer is sometimes called an *adaptive observer* since it provides at the same time estimates of the states and of the parameters.

Separation of Identification and Control

Although we have derived a direct adaptive control scheme, the identifier and the controller can be distinguished. The gains c_0, c, d_0 and d serving to generate u are associated with the controller, while those used to compute e_2 are associated with the identifier. In fact, it is not necessary that these be identical for the identifier error to be as defined in (3.3.19). This is because (3.3.19) was derived using the fundamental identity (3.3.17), which is valid no matter how u is actually computed. In other words, the identifier can be used off-line, without actually updating the controller parameters if necessary. This is also useful, for example, in case of input saturation (cf. Goodwin & Mayne

[1987]). If the actual input to the LTI plant is different from the computed input $u = \theta^T w$ (due to actuator saturation, for example), the identifier will still have consistent input signals, provided that the signal u entering the identifier is the actual input entering the LTI plant.

3.3.2 Output Error Direct Adaptive Control

An output error scheme is based on the output error $e_0 = y_p - y_m$. Note that by applying $\hat{M}\hat{L}$ to both sides of (3.3.17), we find

$$\hat{M}(u) = c_0^* y_p + \hat{M}(\bar{\theta}^{*T}\bar{w}) \tag{3.3.20}$$

As before, the control input u is set equal to $u = \theta^T w$, but now, this equality is used to derive the identifier error equation

$$\begin{aligned} e_0 = y_p - y_m &= \frac{1}{c_0^*}\hat{M}(u - \bar{\theta}^{*T}\bar{w}) - \hat{M}(r) \\ &= \frac{1}{c_0^*}\hat{M}((c_0 - c_0^*)r + (\bar{\theta}^T - \bar{\theta}^{*T})\bar{w}) \\ &= \frac{1}{c_0^*}\hat{M}(\phi^T w) \end{aligned} \tag{3.3.21}$$

which has the form of the *basic* strictly positive real (SPR) error equation of Chapter 2. The gradient identification algorithms of Section 2.6 can therefore be used, provided that $\hat{M}$ is SPR. However, since this requires $\hat{M}$ to have relative degree at most 1, this scheme does not work for plants with relative degree greater than 1.

The approach can however be saved by modifying the scheme, as for example in Narendra, Lin, & Valavani [1980]. We now review their scheme for the case when the high-frequency gain k_p is known, and we let $c_0 = c_0^*$.

The controller structure of the output error scheme is identical to the controller structure of the input error scheme, while the identifier structure is different. It relies on the identifier error

$$e_1 = \frac{1}{c_0^*}\hat{M}\hat{L}(\bar{\phi}^T\bar{v} - \gamma\bar{v}^T\bar{v}e_1) \tag{3.3.22}$$

which is now of the form of the *modified* SPR error equation of Chapter 2. As previously, $\bar{v}$ is identical to v, but with the first component removed. Practically, (3.3.22) is not implemented as such. Instead, we use (3.3.17) to obtain

$$\begin{aligned} e_1 &= \frac{1}{c_0^*}\hat{M}\hat{L}(\bar{\theta}^T\bar{v} - \hat{L}^{-1}(u) + c_0^*(\hat{M}\hat{L})^{-1}(y_p) - \gamma\bar{v}^T\bar{v}e_1) \\ &= y_p - \frac{1}{c_0^*}\hat{M}(u) + \frac{1}{c_0^*}\hat{M}\hat{L}(\bar{\theta}^T\hat{L}^{-1}(\bar{w}) - \gamma\bar{v}^T\bar{v}e_1) \end{aligned} \tag{3.3.23}$$

As before, the control signal is set equal to $u = \theta^T w = c_0^* r + \bar{\theta}^T\bar{w}$, and the equality is used to derive the error equation for the identifier

$$\begin{aligned} e_1 &= y_p - \hat{M}(r) - \frac{1}{c_0^*}\hat{M}(\bar{\theta}^T\bar{w}) + \frac{1}{c_0^*}\hat{M}\hat{L}(\bar{\theta}^T\hat{L}^{-1}(\bar{w}) - \gamma\bar{v}^T\bar{v}e_1) \\ &= y_p - y_m - \frac{1}{c_0^*}\hat{M}\hat{L}((\hat{L}^{-1}\bar{\theta}^T - \bar{\theta}^T\hat{L}^{-1})(\bar{w}) + \gamma\bar{v}^T\bar{v}e_1) \end{aligned} \tag{3.3.24}$$

Again, the identifier error involves the output error $e_0 = y_p - y_m$. The additional term, which appeared starting with the work of Monopoli [1974], is denoted

$$y_a = \frac{1}{c_0^*}\hat{M}\hat{L}((\hat{L}^{-1}\bar{\theta}^T - \bar{\theta}^T\hat{L}^{-1})(\bar{w}) + \gamma\bar{v}^T\bar{v}e_1) \tag{3.3.25}$$

and the resulting error $e_1 = y_p - y_m - y_a$ is called the *augmented error*, in contrast to the original output error $e_0 = y_p - y_m$.

The error (3.3.22) is of the form of the modified SPR error equation of Chapter 2 provided that $\hat{M}\hat{L}$ is a strictly positive real transfer function. If this condition is satisfied, the properties of the identifier will follow and are the basis of the stability proof of Section 3.7.

Assumptions

The algorithm relies on assumptions (A1)–(A3) and the following assumption.

(A5) High-Frequency Gain and SPR Assumptions

Assume that k_p is known and that there exists $\hat{L}^{-1}$, a stable, minimum phase transfer function of relative degree $n - m - 1$, such that $\hat{M}\hat{L}$ is SPR.

The practical implementation of the algorithm is summarized hereafter.

Output Error Direct Adaptive Control Algorithm—Implementation

Assumptions

(A1)–(A3), (A5)

Data

n, m, k_p

Input

$r(t), y_p(t) \in \mathbb{R}$

Output

$u(t) \in \mathbb{R}$

Internal Signals

$\bar{w}(t) \in \mathbb{R}^{2n-1}$ $[w^{(1)}(t), w^{(2)}(t) \in \mathbb{R}^{n-1}]$

$\bar{\theta}(t) \in \mathbb{R}^{2n-1}$ $[(c(t), d(t) \in \mathbb{R}^{n-1}, d_0(t) \in \mathbb{R}]$

$\bar{v}(t) \in \mathbb{R}^{2n-1}$

$e_1(t), y_a(t), y_m(t) \in \mathbb{R}$

Initial conditions are arbitrary.

Design Parameters

Choose

- $\hat{M}$ (i.e. $k_m, \hat{n}_m, \hat{d}_m$) satisfying (A2) and (A5).
- $\Lambda \in \mathbb{R}^{n-1 \times n-1}$, $b_\lambda \in \mathbb{R}^{n-1}$, in controllable canonical form, such that det $(sI - \Lambda)$ is Hurwitz, and contains the zeros of $\hat{n}_m(s)$.
- $\hat{L}^{-1}$ stable, minimum phase transfer function of relative degree $n - m - 1$, such that $\hat{M}\hat{L}$ is SPR.
- $g, \gamma > 0$.

Controller Structure

$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$

$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$

$\bar{\theta}^T = (c^T, d_0, d^T)$

$\bar{w}^T = (w^{(1)^T}, y_p, w^{(2)^T})$

$c_0^* = k_m / k_p > 0$

$u = c_0^* r + \bar{\theta}^T \bar{w}$

Identifier Structure

$\bar{v}^T = \hat{L}^{-1}(\bar{w})$

$y_m = \hat{M}(r)$

$$y_a = \frac{1}{c_0^*} \hat{M}\hat{L}(\hat{L}^{-1}(\bar{\theta}^T \bar{w}) - \bar{\theta}^T \hat{L}^{-1}(\bar{w}) - \gamma \bar{v}^T \bar{v} e_1)$$

$e_1 = y_p - y_m - y_a$

Gradient Algorithm

$\dot{\bar{\theta}} = -g e_1 \bar{v}$

□

Differences Between Input and Output Error

Traditionally, the starting point in the derivation of model reference adaptive control schemes has been the output error $e_0 = y_p - y_m$. Using the error between the plant and the reference model to update controller parameters is intuitive. However, stability proofs suggest that SPR conditions must be satisfied by the model and that an augmented error should be used when the relative degree of the plant is greater than 1. The derivation of the input error scheme shows that model reference adaptive control can in fact be achieved *without* formally involving the output error and without SPR conditions on the reference model.

Important differences should be noted between the input and output error schemes. The first is that the derivation of the equation error (3.3.24) from (3.3.22) relies on the input signal u being equal to the computed value $u = \theta^T w$, at all times. If the input saturates, updates of the identifier will be erroneous. When the input error scheme is used, this problem can be avoided, provided that the actual input entering the LTI plant is available and used in the identifier. This is because (3.3.19) is based on (3.3.17) and does not assume any particular value of u. If needed, the parameters used for identification and control can also be separated, and the identifier can be used "off-line."

A second difference appears between the input and output error schemes when the high-frequency gain k_p is unknown, and the relative degree of the plant is greater than 1. The error e_1 derived in (3.3.22) is not implementable if c_0^* is unknown. Although an SPR error equation can still be obtained in the unknown high-frequency gain case, the solution proposed by Morse [1980] (and also Narendra, Lin, & Valavani [1980]) requires an overparameterization of the identifier which excludes the possibility of asymptotic stability even when persistency of excitation (PE) conditions are satisfied (cf. Boyd & Sastry [1986], Anderson, Dasgupta, & Tsoi [1985]). In view of the recent examples due to Rohrs, and the connections between exponential convergence and robustness (see Chapter 5), this appears to be a major drawback of the algorithm.

Another advantage of the input error scheme is to lead to a linear error equation for which other identification algorithms, such as the least-squares algorithm, are available. These algorithms are an advantageous alternative to the gradient algorithm. Further, it was shown recently that there are advantages of the input error scheme in terms of robustness to unmodeled dynamics (cf. Bodson [1988]).

In some cases, the input error scheme requires more computations. This is because the observers for $w^{(1)}$, $w^{(2)}$ are on order n, instead of $n-1$ for the output error scheme. Also, the filter $\hat{L}^{-1}$ is one order higher. When the relative degree is 1, significant simplifications arise in the output error scheme, as discussed now.

Output Error Direct Adaptive Control—The Relative Degree 1 Case

The condition that $\hat{M}\hat{L}$ be SPR is considerably stronger than the condition that $\hat{M}\hat{L}$ simply be invertible (as required by the input error scheme and guaranteed by (A2)). The relative degree of $\hat{L}^{-1}$, however, is only required to be $n-m-1$, as compared to $n-m$ for proper invertibility. In the case when the relative degree $n-m$ of the model and of the plant is 1, $\hat{L}^{-1}$ is unnecessary along with the additional signal y_a. The output error direct adaptive control scheme then has a much simpler form, in which the error equation used for identification involves the output error $e_0 = y_p - y_m$ only. The simplicity of this scheme makes it attractive in that case. We assume therefore the following:

(A6) Relative Degree 1 and SPR Assumptions

$n - m = 1$, $\hat{M}$ is SPR.

Output Error Direct Adaptive Control Algorithm, Relative Degree 1—Implementation

Assumptions

(A1)–(A3), (A6)

Data

n, k_p

Input

$r(t), y_p(t) \in \mathbb{R}$

Output

$u(t) \in \mathbb{R}$

Internal Signals

$w(t) \in \mathbb{R}^{2n}$ $[w^{(1)}(t), w^{(2)}(t) \in \mathbb{R}^{n-1}]$

$\theta(t) \in \mathbb{R}^{2n}$ $[c_0(t), d_0(t) \in \mathbb{R}, c(t), d(t) \in \mathbb{R}^{n-1}]$

$y_m(t), e_0(t) \in \mathbb{R}$

Initial conditions are arbitrary.

Design Parameters

Choose

- $\hat{M}$ (i.e. k_m, $\hat{n}_m$, $\hat{d}_m$) satisfying (A2) and (A6).
- $\Lambda \in \mathbb{R}^{n-1 \times n-1}$, $b_\lambda \in \mathbb{R}^{n-1}$ in controllable canonical form and such that $\det(sI - \Lambda) = \hat{n}_m(s)$.
- $g > 0$.

Controller Structure

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$
$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$
$$\theta^T = (c_0, c^T, d_0, d^T)$$
$$w^T = (r, w^{(1)^T}, y_p, w^{(2)^T})$$
$$u = \theta^T w$$

Identifier Structure

$$y_m = \hat{M}(r)$$
$$e_0 = y_p - y_m$$

Gradient Algorithm

$$\dot{\theta} = -g e_0 w$$

□

Comment

The identifier error equation is (3.3.24) and is the *basic* SPR error equation of Chapter 2. The high-frequency gain k_p (and consequently c_0) can be assumed to be unknown, but the sign of k_p must still be known to ensure that $c_0^* > 0$, so that $(1/c_0^*)\hat{M}$ is SPR.

3.3.3 Indirect Adaptive Control

In the indirect adaptive control scheme presented in this section, estimates of the plant parameters k_p, $\hat{n}_p$, and $\hat{d}_p$ are obtained using the standard equation error identifier of Chapter 2. The controller parameters c_0, $\hat{c}$ and $\hat{d}$ are then computed using the relationships resulting from the matching equality (3.2.6).

Note that the dimension of the signals $w^{(1)}$, $w^{(2)}$ used for identification in Chapter 2 is n, the order of the plant. For control, it is sufficient that this dimension be $n-1$. However, in order to share the observers for identification and control, we will let their dimension be n. Proposition 3.2.1 is still true then, but the degrees of the polynomials become, respectively, $\partial\hat{\lambda} = n$, $\partial\hat{\lambda}_0 = n - m$, $\partial\hat{q} = n - m$, $\partial\hat{d} = n - 1$, and $\partial\hat{c} = n - 1$. Since $\partial\hat{d} = n - 1$, it can be realized as $d^T(sI - \Lambda)^{-1}b$, without the direct gain d_0 from y_p. This a (minor) technical difference, and for simplicity, we will keep our previous notation. Thus, we define

$$\bar{\theta}^T := (c^T, d^T) \in \mathbb{R}^{2n} \qquad \bar{w}^T := (w^{(1)^T}, w^{(2)^T}) \in \mathbb{R}^{2n} \tag{3.3.26}$$

and

$$\theta^T := (c_0, \bar{\theta}^T) \in \mathbb{R}^{2n+1} \qquad w^T := (r, \bar{w}^T) \in \mathbb{R}^{2n+1} \tag{3.3.27}$$

The controller structure is otherwise completely identical to the controller structure described previously.

The identifier parameter is now different from the controller parameter θ. We will denote, in analogy with (2.2.17)

$$\begin{aligned}\pi^T &:= (a^T, b^T)\\ &:= (a_1, \ldots, a_{m+1}, 0, \ldots, b_1, \ldots, b_n) \in \mathbb{R}^{2n}\end{aligned} \tag{3.3.28}$$

Since the relative degree is assumed to be known, there is no need to update the parameters a_{m+2}—so that we let these parameters be zero in (3.3.28). The corresponding components of $\bar{w}$ are thus not used for identification. We let $\tilde{w}$ be equal to $\bar{w}$ except for those components which are not used and are thus set to zero, so that

$$\tilde{w}^T := (w_1^{(1)}, \ldots, w_{m+1}^{(1)}, 0, \ldots, w^{(2)^T}) \in \mathbb{R}^{2n} \tag{3.3.29}$$

A consequence (that will be used in the stability proof in Section 3.7) is that the relative degree of the transfer function from $u \to \tilde{w}$ is at least $n - m$.

The nominal value of the identifier parameter π^* can be found from the results of Chapter 2 through the polynomial equalities in (2.2.5), that is

$$\begin{aligned}\hat{a}^*(s) &= a_1^* + a_2^* s + \cdots + a_{m+1}^* s^m = k_p \hat{n}_p(s)\\ \hat{b}^*(s) &= b_1^* + b_2^* s + \cdots + b_n^* s^{n-1} = \hat{\lambda}(s) - \hat{d}_p(s)\end{aligned} \tag{3.3.30}$$

The *identifier parameter error* is now denoted

$$\psi := \pi - \pi^* \in \mathbb{R}^{2n} \tag{3.3.31}$$

The transformation $\pi \to \theta$ is chosen following a certainty equivalence principle to be the same as the transformation $\pi^* \to \theta^*$, as in (3.2.7)–(3.2.9). Note that our estimate of the high-frequency gain k_p is

a_{m+1}. Since $c_0^* = k_m / k_p$, we will let $c_0 = k_m / a_{m+1}$. The control input u will be unbounded if a_{m+1} goes to zero, and to avoid this problem, we make the following assumption.

(A7) Bound on the High-Frequency Gain

Assume $k_p \geq k_{\min} > 0$.

The practical implementation of the indirect adaptive control algorithm is summarized hereafter.

Indirect Adaptive Control Algorithm—Implementation

Assumptions

(A1)–(A3), (A7)

Data

$n, m, k_{\min}$

Input

$r(t), y_p(t) \in \mathbb{R}$

Output

$u(t) \in \mathbb{R}$

Internal Signals

$w(t) \in \mathbb{R}^{2n+1}$ $[w^{(1)}(t), w^{(2)}(t) \in \mathbb{R}^n]$

$\theta(t) \in \mathbb{R}^{2n+1}$ $[c_0(t) \in \mathbb{R}, c(t), d(t) \in \mathbb{R}^n]$

$\pi(t) \in \mathbb{R}^{2n}$ $[a(t), b(t) \in \mathbb{R}^n]$

$\tilde{w}(t) \in \mathbb{R}^{2n}$

$y_i(t), e_3(t) \in \mathbb{R}$

Initial conditions are arbitrary, except $a_{m+1}(0) > k_{\min}$.

Design Parameters

Choose

- $\hat{M}$ (i.e. k_m, $\hat{n}_m$, $\hat{d}_m$) satisfying (A2).
- $\Lambda \in \mathbb{R}^{n \times n}$, $b_\lambda \in \mathbb{R}^n$ in controllable canonical form, such that det $(sI - \Lambda) = \hat{\lambda}(s)$ is Hurwitz and $\hat{\lambda}(s) = \hat{\lambda}_0(s)\hat{n}_m(s)$.
- $g, \gamma > 0$.

Controller Structure

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$\theta^T = (c_0, c^T, d^T) = (c_0, c_1, \ldots, c_n, d_1, \ldots, d_n)$$

$w^T = (r, w^{(1)^T}, w^{(2)^T})$

$u = \theta^T w$

Identifier Structure

$\pi^T = (a^T, b^T) = (a_1, \ldots, a_{m+1}, 0, \ldots, b_1, \ldots, b_n)$

$\tilde{w} = (w_1^{(1)}, \ldots, w_{m+1}^{(1)}, 0, \ldots, w^{(2)^T})$

$y_i = \pi^T \tilde{w}$

$e_3 = \pi^T \tilde{w} - y_p$

Normalized Gradient Algorithm with Projection

$$\dot{\pi} = -g \frac{e_3 \tilde{w}}{1 + \gamma \tilde{w}^T \tilde{w}}$$

If $a_{m+1} = k_{\min}$ and $\dot{a}_{m+1} < 0$, then let $\dot{a}_{m+1} = 0$.

Transformation Identifier Parameter → Controller Parameter

Let the polynomials with time-varying coefficients

$\hat{a}(s) = a_1 + \cdots + a_{m+1} s^m \qquad \hat{c}(s) = c_1 + \cdots + c_n s^{n-1}$

$\hat{b}(s) = b_1 + \cdots + b_n s^{n-1} \qquad \hat{d}(s) = d_1 + \cdots + d_n s^{n-1}$

Divide $\hat{\lambda}_0 \hat{d}_m$ by $(\hat{\lambda} - \hat{b})$, and let $\hat{q}$ be the quotient.

θ is given by the coefficients of the polynomials

$$\hat{c} = \hat{\lambda} - \frac{1}{a_{m+1}} \hat{q}\, \hat{a}$$

$$\hat{d} = \frac{1}{a_{m+1}} (\hat{q}\, \hat{\lambda} - \hat{q}\, \hat{b} - \hat{\lambda}_0 \hat{d}_m)$$

and by

$$c_0 = \frac{k_m}{a_{m+1}}$$

□

Transformation Identifier Parameter → Controller Parameter

We assumed that the transformation from the identifier parameter π to the controller parameter θ is performed *instantaneously*. Note that $\hat{\lambda} - \hat{b}$ is a monic polynomial, so that $\hat{q}$ is also a monic polynomial (of degree $n - m$). Its coefficients can be expressed as the sum of products of coefficients of $\hat{\lambda}_0 \hat{d}_m$ and $\hat{\lambda} - \hat{b}$. The same is true for $\hat{c}$, $\hat{d}$, and c_0 with an additional division by a_{m+1}. Therefore, given n and m, the transformation consists of a fixed number of multiplications, additions, and a division.

Note also that if the coefficients of $\hat{a}$ and $\hat{b}$ are bounded, and if a_{m+1} is bounded away from zero (as is guaranteed by the projection), then the coefficients of $\hat{q}$, $\hat{c}$, $\hat{d}$, and c_0 are bounded. Therefore, the transformation is also continuously differentiable and has bounded derivatives.

3.3.4 Alternate Model Reference Schemes

The input error scheme is closely related to the schemes presented in discrete time by Goodwin & Sin [1984], and in continuous time by Goodwin & Mayne [1987]. Their identifier structure is identical to the structure used here, but their controller structure is somewhat different. In our notation, Goodwin & Mayne choose

$$\hat{M}(s) = k_m \frac{\hat{n}(s)}{\hat{\lambda}(s) \hat{L}(s)} \tag{3.3.32}$$

where $\hat{n}$, $\hat{\lambda}$ and $\hat{L}$ are *polynomials* of degree $\leq n$, n, and $n - m$, respectively. The polynomials $\hat{\lambda}$, $\hat{L}$ are used for similar purposes as in the input error scheme. However, except for possible pole-zero cancellations, $\hat{\lambda}\hat{L}$ now also defines the model poles in (3.3.32). The filtered reference input

$$\bar{r} = k_m \frac{\hat{n}(s)}{\hat{\lambda}(s)} (r) \tag{3.3.33}$$

is used as input to the actual controller. Then, the transfer function $\bar{r} \to y_p$ is made to match $\hat{L}^{-1}$, so that the transfer function from $r \to y_p$ is $\hat{M}$. Thus, by prefiltering the input, the control problem of matching a transfer function $\hat{M}$ is altered to the problem of matching the arbitrary all-pole transfer function $\hat{L}^{-1}$.

The input error adaptive control scheme of Section 3.3.1 can be used to achieve this new objective and is represented in Figure 3.6. This scheme is the one obtained by Goodwin & Mayne (up to a small remaining difference described hereafter). Since the new model is $\hat{L}^{-1}$, the new transfer function $\hat{M}\hat{L}$ is equal to 1. Note that, in this instance, the input and output errors are identical and the input and output error schemes are very similar. The analysis is also considerably simplified.

Goodwin & Mayne's algorithms essentially control the plant by reducing the transfer function to an all-pole transfer function of relative degree $n - m$. The additional dynamics are provided by prefiltering the reference input. Thus, the input error scheme presented in Section 3.3.1 is a more general scheme, allowing for the placement of all the closed-

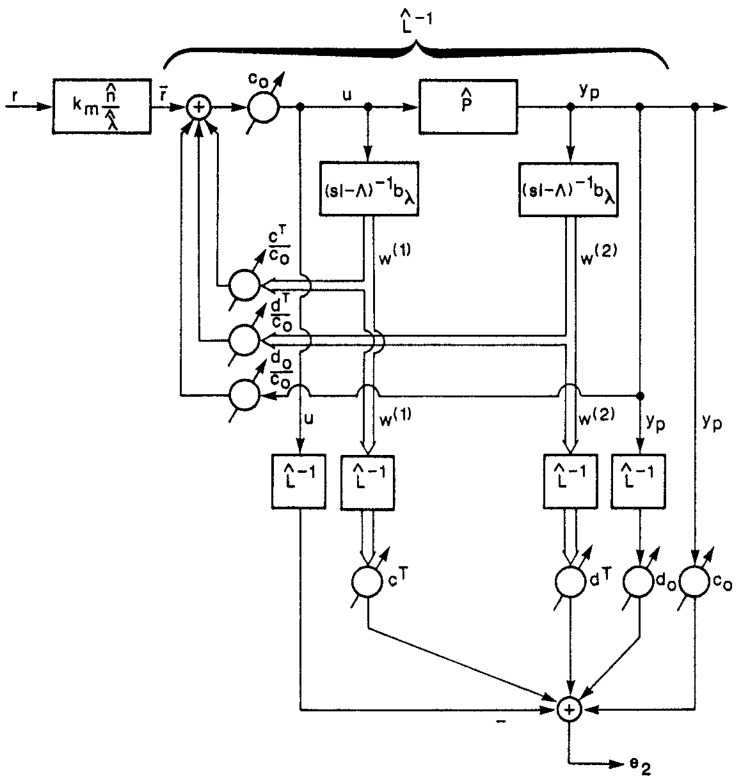

Figure 3.6: Alternate Input Error Scheme

loop poles directly at the desired locations without prefiltering.

Note that since identification and control can be separated in the input error scheme, we may identify $1/c_0$ and $\bar{\theta}/c_0$ rather than c_0 and $\bar{\theta}$. This is also shown in Figure 3.6. By dividing the identifier error e_2 by c_0, the appropriate linear error equation may be found and used for identification.

It is curious to note that the problems encountered are different depending whether we identify c_0 or $1/c_0$. If we identify $1/c_0$, as we did in the indirect scheme, the control input $u = c_0 r + \bar{\theta}^T \bar{w}$ will be unbounded if the estimate of $1/c_0$ goes to zero. To avoid the zero crossing, we require knowledge of the sign of $1/c_0$ (that is, of k_p), and of a lower bound on $1/c_0$, that is a *lower* bound on k_p to be used with the

projection algorithm.

If we identify c_0 directly, as we did in the input error scheme, a different problem appears. If $c_0 = 0$ and $\bar{\theta} = 0$, then $u = 0$ and $e_2 = 0$ (cf. Figure 3.5). No adaptation will occur ($\dot{\phi} = 0$), although $y_p - y_m$ does not tend necessarily to zero and may even be unbounded. This is an identification problem, since we basically lose information in the regression vector. To avoid it, we require the knowledge of the sign of c_0 (i.e., of k_p) and a lower bound on c_0, therefore an *upper* bound on k_p, to be used by the projection algorithm.

3.3.5 Adaptive Pole Placement Control

The model reference adaptive control approach requires a minimum phase assumption on the plant $\hat{P}(s)$. This results from the necessity to cancel the plant zeros in order to replace them by the model zeros. One might consider the approach of letting the model be

$$\hat{M}(s) = k_m \frac{\hat{n}_p(s)}{\hat{d}_m(s)} \qquad i.e. \qquad \hat{n}_m(s) = \hat{n}_p(s) \tag{3.3.34}$$

that is, require that only the closed-loop poles be assigned. An adaptive control based on this idea is called an *adaptive pole placement control* algorithm. We now discuss some of the differences between this and the model reference adaptive control algorithms presented above.

First note that in (3.3.34), the reference model itself becomes adaptive, with $\hat{n}_p$ replaced by its estimated value. Since $\hat{n}_p$ is unknown *a priori* and it is not Hurwitz, it is impossible to choose $\hat{\lambda} = \hat{\lambda}_0 \hat{n}_m$ as before. Now, let $\hat{\lambda}$ be an arbitrary Hurwitz polynomial, and consider the same controller as previously, so that (3.2.4) is valid. The nominal values c_0^*, $\hat{c}^*$, $\hat{d}^*$ such that the closed-loop transfer function is equal to the $\hat{M}(s)$ defined in (3.3.34) must satisfy

$$(\hat{\lambda} - \hat{c}^*)\hat{d}_p - k_p \hat{n}_p \hat{d}^* = \left[c_0^* \frac{k_p}{k_m} \right] \hat{\lambda}\, \hat{d}_m \tag{3.3.35}$$

This equation is a *Diophantine equation*, that is a polynomial equation of the form

$$\hat{a}\hat{x} + \hat{b}\hat{y} = \hat{c} \tag{3.3.36}$$

A necessary condition for a solution $\hat{x}, \hat{y}$ to exist is that any common zero of $\hat{a}, \hat{b}$ is also a zero of $\hat{c}$. A sufficient condition is simply that $\hat{a}, \hat{b}$ be coprime, in this case, $\hat{n}_p$, $\hat{d}_p$ coprime (see lemma A6.2.3 in the

Appendix for the general solution in that case). Previously, (3.3.35) was replaced by (3.2.6), with $\hat{n}_p$ appearing on the right hand side, so that any common zero of $\hat{n}_p$, $\hat{d}_p$ was automatically a zero of the right hand side. Therefore, the solution always existed, although not unique when $\hat{n}_p$, $\hat{d}_p$ were not coprime.

An indirect adaptive pole placement control algorithm may be obtained in a similar way as the indirect model reference adaptive control algorithm of Section 3.3.3. Then $\hat{c}, \hat{d}$ are obtained by solving (3.3.35) with k_p, $\hat{n}_p$, $\hat{d}_p$ replaced by their estimates. A difficulty arises to guarantee that the estimates of $\hat{n}_p \hat{d}_p$ are coprime, since the solution of (3.3.35) will usually not exist otherwise. This was not necessary in the model reference case where the solution always existed.

Proving stability for an adaptive pole placement algorithm is somewhat complicated. It is often assumed that the input is sufficiently rich and that some procedure guarantees coprimeness of the estimates. Nevertheless, these algorithms have the significant advantage of not requiring minimum phase assumptions. A further discussion is presented in Section 6.2.2.

3.4 THE STABILITY PROBLEM IN ADAPTIVE CONTROL

Stability Definitions

Various definitions and concepts of stability have been proposed. A classical definition for systems of the form

$$\dot{x} = f(t,x) \tag{3.4.1}$$

is the stability in the sense of Lyapunov defined in Chapter 1.

The adaptive systems described so far are of the special form

$$\dot{x} = f(t,x,r(t)) \tag{3.4.2}$$

where r is the input to the system and x is the overall state of the system, including the plant, the controller, and the identifier. For practical reasons, stability in the sense of Lyapunov is not sufficient for adaptive systems. As we recall, this definition is a local property, guaranteeing that the trajectories will remain arbitrarily close to the equilibrium, *when started sufficiently close*. In adaptive systems, we do not have any control on how close initial conditions are to equilibrium values. A natural stability concept is then the *bounded-input bounded-state* stability (BIBS): for any $r(.)$ bounded, and $x_0 \in \mathbb{R}^n$, the solution $x(.)$ remains bounded. This is the concept of stability that will be used in this chapter.

The Problem of Proving Stability in Adaptive Control

The stability of the identifiers presented in Chapter 2 was assessed in theorem 2.4.5. There, the stability of the plant was assumed explicitly. In adaptive control, the stability of the controlled plant must be guaranteed by the identifier, which seriously complicates the problem. The stability of the *overall adaptive system*, which includes the plant, the controller, and the identifier, must then be considered.

To understand the nature of the problem, we will take a general approach in this section and consider the generic model reference adaptive control system shown in Figure 3.7.

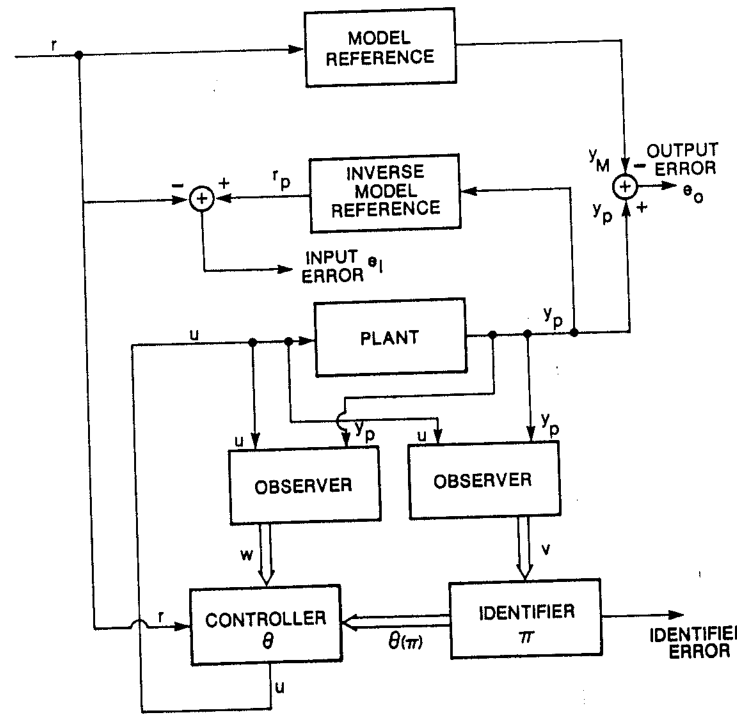

Figure 3.7: Generic Model Reference Adaptive Control System

The signals and systems defined previously can be recognized. θ is the controller parameter, and π is the identifier parameter. In the case of direct control, $\theta = \pi$, that is, the parameter being identified is directly the controller parameter. The identifier error may be the output error $e_0 = y_p - y_m$, the input error $e_i = r_p - r$, or any other error used for identification.

The problem of stability can be understood as follows. Intuitively, the plant with the control loop will be stable if θ is sufficiently close to the true value θ^*. However, as we saw in Chapter 2, the convergence of the identifier is dependent on the stability and persistent excitation of signals originating from the control loop.

To break this circular argument, we must first express properties of the identifier that are independent of the stability and persistency of excitation of these signals. Such properties were already derived in Chapter 2, and were expressed in terms of the identifier error. Recall that the identifier *parameter* error $\pi - \pi^*$ does not converge to zero but that only the identifier error converges to zero in some sense. Thus, we cannot argue that for t sufficiently large, the controller parameter θ will be arbitrarily close to the nominal value that stabilizes the plant-control loop.

Instead of relying on the convergence of θ to θ^* to prove stability, we can express the control signal as a nominal control signal—that makes the controlled plant match the reference model—plus a control error. The problem then is to transfer the properties of the identifier to the control loop, that is, the identifier error to the control error, and prove stability. Several difficulties are encountered here. First, the transformation $\theta(\pi)$ is usually nonlinear. In direct adaptive control, the transformation is the identity, and the proof is consequently simplified. Another difficulty arises however from the different signals v and w used for identification and control. A major step will be to transfer properties of the identifier involving v to properties of the controller involving w. Provided that the resulting control error is a "small" gain from plant signals, the proof of stability will basically be a *small gain theorem* type of proof, a generic proof to assess the stability of nonlinear time varying systems (cf. Desoer & Vidyasagar [1975]).

3.5 ANALYSIS OF THE MODEL REFERENCE ADAPTIVE CONTROL SYSTEM

We now return to the model reference adaptive control system presented in Sections 3.1–3.3. The results derived in this section are the basis for analyses presented in this and following chapters. Many identities involve signals which are not available in practice (since $\hat{P}$ is unknown) but are well defined for the analysis. Most results also rely on the control input being defined by

$$u = \theta^T w$$

$$\theta^T = (c_0, c^T, d_0, d^T)$$
$$w^T = (r, w^{(1)^T}, y_p, w^{(2)^T}) \tag{3.5.1}$$

Error Formulation

It will be useful to represent the adaptive system in terms of its deviation with respect to the ideal situation when $\theta = \theta^*$, that is, $\phi = 0$. This step is similar to transferring the equilibrium point of a differential equation as (3.4.1) to $x = 0$ by a change of coordinates.

Recall that we defined r_p in (3.3.1) as

$$r_p = \hat{M}^{-1}(y_p) \tag{3.5.2}$$

while

$$y_m = \hat{M}(r) \tag{3.5.3}$$

Applying $\hat{L}$ to (3.3.10), it follows, since θ^* is constant, that

$$u = c_0^* r_p + \bar{\theta}^{*T} \bar{w} \tag{3.5.4}$$

and, since u is given by (3.5.1),

$$r_p = r + \frac{1}{c_0^*} \phi^T w \tag{3.5.5}$$

Further, applying $\hat{M}$ to both sides of (3.5.5)

$$y_p = y_m + \frac{1}{c_0^*} \hat{M}(\phi^T w) \tag{3.5.6}$$

The signal $\phi^T w$ will be called the *control error*. We note that the *input error* $e_i = r_p - r$ is directly proportional to the *control error* $\phi^T w$ (cf. (3.5.5)), while the *output error* $e_0 = y_p - y_m$ is related to the control error through the model transfer function $\hat{M}$ (cf. (3.5.6)).

Since $y_p = \hat{P}(u) = \hat{M}(r_p)$, the control input can also be expressed in terms of the control error as

$$u = \hat{P}^{-1}\hat{M}(r_p) = \hat{P}^{-1}\hat{M}(r + \frac{1}{c_0^*} \phi^T w) \tag{3.5.7}$$

and the vector $\bar{w}$ is similarly expressed as

$$\bar{w} = \begin{bmatrix} w^{(1)} \\ y_p \\ w^{(2)} \end{bmatrix} = \begin{bmatrix} (sI-\Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI-\Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} (r + \frac{1}{c_0^*} \phi^T w) \tag{3.5.8}$$

while v (cf. (3.3.16)) is given by

$$v = \hat{L}^{-1} \begin{bmatrix} 1 \\ (sI-\Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI-\Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} (r + \frac{1}{c_0^*} \phi^T w) \tag{3.5.9}$$

For the purpose of the analysis, we also define (cf. (3.3.10))

$$z := \hat{L}(v) = \begin{bmatrix} r_p \\ \bar{w} \end{bmatrix} = \begin{bmatrix} 1 \\ (sI-\Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI-\Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} (r + \frac{1}{c_0^*} \phi^T w) \tag{3.5.10}$$

Note that the transfer functions appearing in (3.5.6)–(3.5.10) are all stable (using assumptions (A1)–(A2) and the definitions of Λ and $\hat{L}^{-1}$).

Model Signals

The *model signals* are defined as the signals corresponding to the plant signals when $\theta = \theta^*$, that is, $\phi = 0$. As expected, the model signals corresponding to y_p and r_p are y_m and r, respectively (cf. (3.5.6) and (3.5.5)). Similarly, we define

$$\bar{w}_m := \begin{bmatrix} w_m^{(1)} \\ y_m \\ w_m^{(2)} \end{bmatrix} := \begin{bmatrix} (sI-\Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI-\Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} (r)$$

$$:= \hat{H}_{\bar{w}_m r}(r) \tag{3.5.11}$$

and

$$v_m := \hat{L}^{-1}(z_m) := \hat{L}^{-1} \begin{bmatrix} 1 \\ (sI-\Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI-\Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} (r) \tag{3.5.12}$$

By defining

$$w_m := \begin{bmatrix} r \\ \bar{w}_m \end{bmatrix} \tag{3.5.13}$$

we note the remarkable fact that

$$w_m = z_m \tag{3.5.14}$$

Since the transfer functions relating r to the model signals are all stable, and since r is bounded (assumption (A3)), it follows that all model signals are bounded functions of time. Consequently, if the differences between plant and model signals are bounded, the plant signals will be bounded.

State-Space Description

We now show how a state-space description of the overall adaptive system can be obtained. In particular, we will check that no cancellation of possibly unstable modes occurs when $\theta = \theta^*$.

The plant has a minimal state-space representation $[A_p, b_p, c_p^T]$ such that

$$\hat{P}(s) = k_p \frac{\hat{n}_p(s)}{\hat{d}_p(s)} = c_p^T (sI - A_p)^{-1} b_p \tag{3.5.15}$$

With the definitions of $w^{(1)}$, $w^{(2)}$ in (3.2.17)–(3.2.19), the plant with observer is described by

$$\begin{aligned} \dot{x}_p &= A_p x_p + b_p u \\ \dot{w}^{(1)} &= \Lambda w^{(1)} + b_\lambda u \\ \dot{w}^{(2)} &= \Lambda w^{(2)} + b_\lambda y_p = \Lambda w^{(2)} + b_\lambda c_p^T x_p \end{aligned} \tag{3.5.16}$$

The control input u can be expressed in terms of its desired value, plus the control error $\phi^T w$, as

$$u = \theta^T w = \theta^{*T} w + \phi^T w \tag{3.5.17}$$

so that

$$\begin{bmatrix} \dot{x}_p \\ \dot{w}^{(1)} \\ \dot{w}^{(2)} \end{bmatrix} = \begin{bmatrix} A_p + b_p d_0^* c_p^T & b_p c^{*T} & b_p d^{*T} \\ b_\lambda d_0^* c_p^T & \Lambda + b_\lambda c^{*T} & b_\lambda d^{*T} \\ b_\lambda c_p^T & 0 & \Lambda \end{bmatrix} \cdot \begin{bmatrix} x_p \\ w^{(1)} \\ w^{(2)} \end{bmatrix}$$

$$
+ \begin{bmatrix} b_p \\ b_\lambda \\ 0 \end{bmatrix} \phi^T w + \begin{bmatrix} b_p \\ b_\lambda \\ 0 \end{bmatrix} c_0^* r
$$

$$
y_p = c_p^T x_p \tag{3.5.18}
$$

Defining $x_{pw} \in \mathbb{R}^{3n-2}$ to be the total state of the plant and observer, this equation is rewritten as

$$
\begin{aligned}
\dot{x}_{pw} &= A_m x_{pw} + b_m \phi^T w + b_m c_0^* r \\
y_p &= c_m^T x_{pw}
\end{aligned} \tag{3.5.19}
$$

where $A_m \in \mathbb{R}^{3n-2 \times 3n-2}$, $b_m \in \mathbb{R}^{3n-2}$ and $c_m \in \mathbb{R}^{3n-2}$ are defined through (3.5.18). Since the transfer function from $r \to y_p$ is $\hat{M}$ when $\phi = 0$, we must have that $c_m^T (sI - A_m)^{-1} b_m = (1/c_0^*) \hat{M}(s)$, that is, that $[A_m, b_m, c_m^T]$ is a representation of the model transfer function $\hat{M}$, divided by c_0^*. Therefore, we can also represent the model and its output by

$$
\begin{aligned}
\dot{x}_m &= A_m x_m + b_m c_0^* r \\
y_m &= c_m^T x_m
\end{aligned} \tag{3.5.20}
$$

Note that although the transfer function $\hat{M}$ is stable, its representation is non-minimal, since the order of $\hat{M}$ is n, while the dimension of A_m is $3n - 2$. We can find where the additional modes are located by noting that the representation of the model is that of Figure 3.8. Using standard transfer function manipulations, but avoiding cancellations, we get

$$
\begin{aligned}
c_m^T (sI - A_m)^{-1} b_m &= \frac{\dfrac{k_p \hat{n}_p \hat{\lambda}}{(\hat{\lambda} - \hat{c}^*) \hat{d}_p}}{1 - \dfrac{k_p \hat{n}_p \hat{\lambda}}{(\hat{\lambda} - \hat{c}^*) \hat{d}_p} \dfrac{\hat{d}^*}{\hat{\lambda}}} \\
&= \frac{k_p \hat{\lambda} \hat{n}_p \hat{\lambda}_0 \hat{n}_m}{\hat{\lambda}((\hat{\lambda} - \hat{c}^*) \hat{d}_p - k_p \hat{n}_p \hat{d}^*)}
\end{aligned} \tag{3.5.21}
$$

and, using the matching equality (3.2.6)

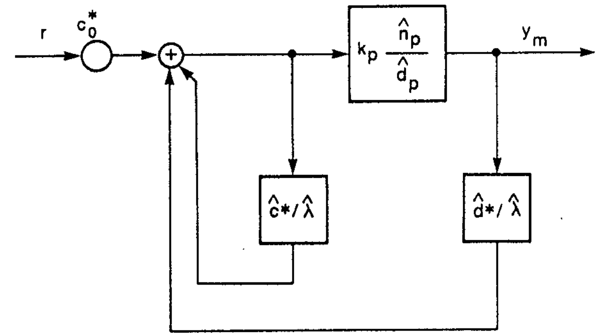

Figure 3.8: Representation of the Reference Model

$$
c_m^T (sI - A_m)^{-1} b_m = \frac{1}{c_0^*} k_m \frac{\hat{n}_m}{\hat{d}_m} \frac{\hat{\lambda} \hat{\lambda}_0 \hat{n}_p}{\hat{\lambda} \hat{\lambda}_0 \hat{n}_p} = \frac{1}{c_0^*} \hat{M} \tag{3.5.22}
$$

Thus, the additional modes are those of $\hat{\lambda}$, $\hat{\lambda}_0$, and $\hat{n}_p$, which are all stable by choice of $\hat{\lambda}$, $\hat{\lambda}_0$, and by assumption (A1). In other words, A_m is a stable matrix.

Since r is assumed to be bounded and A_m is stable, the state vector trajectory x_m is bounded. We can represent the plant states as their differences from the model states, letting the *state error* $e = x_{pw} - x_m \in \mathbb{R}^{3n-2}$, so that

$$
\begin{aligned}
\dot{e} &= A_m e + b_m \phi^T w \\
e_0 &= y_p - y_m = c_m^T e
\end{aligned} \tag{3.5.23}
$$

and

$$
e_0 = y_p - y_m = \frac{1}{c_0^*} \hat{M}(\phi^T w) = \hat{M}(\frac{1}{c_0^*} \phi^T w) \tag{3.5.24}
$$

which is equation (3.5.6), (derived above) through a somewhat shorter path.

Note that (3.5.23) is *not* a linear differential equation representing the plant with controller, because w depends on e. This can be resolved by expressing the dependence of w on e as

$$
w = w_m + Qe \tag{3.5.25}
$$

where

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ c_p^T & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\in \begin{bmatrix} \mathbb{R}^{1\times n} & \mathbb{R}^{1\times n-1} & \mathbb{R}^{1\times n-1} \\ \mathbb{R}^{n-1\times n} & \mathbb{R}^{n-1\times n-1} & \mathbb{R}^{n-1\times n-1} \\ \mathbb{R}^{1\times n} & \mathbb{R}^{1\times n-1} & \mathbb{R}^{1\times n-1} \\ \mathbb{R}^{n-1\times n} & \mathbb{R}^{n-1\times n-1} & \mathbb{R}^{n-1\times n-1} \end{bmatrix} = \mathbb{R}^{2n\times 3n-2} \qquad (3.5.26)$$

A differential equation representing the plant with controller is then

$$\dot{e} = A_m e + b_m \phi^T w_m + b_m \phi^T Q e$$
$$e_0 = c_m^T e \qquad (3.5.27)$$

where w_m is an exogeneous, bounded input.

Complete Description—Output Error, Relative Degree 1 Case

To describe the adaptive system completely, one must simply add to this set of differential equations the set corresponding to the identifier. For example, in the case of the output error adaptive control scheme for relative degree 1 plants, the overall adaptive system (including the plant, controller and identifier) is described by

$$\dot{e} = A_m e + b_m \phi^T w_m + b_m \phi^T Q e$$
$$\dot{\phi} = -g c_m^T e w_m - g c_m^T e Q e \qquad (3.5.28)$$

As for most adaptive control schemes presented in this book, the adaptive control scheme is described by a nonlinear, time varying, ordinary differential equation. This specific case (3.5.28) will be used in subsequent chapters as a convenient example.

3.6 USEFUL LEMMAS

The following lemmas are useful to prove the stability of adaptive control schemes. Most lemmas are inspired from lemmas that are present in one form or another in existing stability proofs. In contrast with Sastry [1984] and Narendra, Annaswamy, & Singh [1985], we do not use any ordering of signals (order relations $o(.)$ and $O(.)$), but keep relationships between signals in terms of norm inequalities.

The systems considered in this section are of the general form

$$y = H(u) \qquad (3.6.1)$$

where $H : L_{pe} \to L_{pe}$ is a SISO *causal* operator, that is, such that

$$y_t = (H(u_t))_t \qquad (3.6.2)$$

for all $u \in L_{pe}$ and for all $t \geq 0$. Lemmas 3.6.1–3.6.5 further restrict the attention to LTI systems with proper transfer functions $\hat{H}(s)$.

Lemma 3.6.1 is a standard result in linear system theory and relates the L_p norm of the output to the L_p norm of the input.

Lemma 3.6.1 Input/Output Lp Stability

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function. Let h be the impulse response corresponding to $\hat{H}$.

If $\hat{H}$ is stable

Then for all $p \in [1,\infty]$ and for all $u \in L_p$

$$\| y \|_p \leq \| h \|_1 \| u \|_p + \| \epsilon \|_p \qquad (3.6.3)$$

for all $u \in L_{\infty e}$

$$| y(t) | \leq \| h \|_1 \| u_t \|_\infty + | \epsilon(t) | \qquad (3.6.4)$$

for all $t \geq 0$, where $\epsilon(t)$ is an exponentially decaying term due to the initial conditions.

Proof of Lemma 3.6.1 cf. Desoer & Vidyasagar [1975], p. 241.

It is useful, although not standard, to obtain a result that is the converse of lemma 3.6.1, that is, with u and y interchanged in (3.6.3)–(3.6.4). Such a lemma can be found in Narendra, Lin, & Valavani [1980], Narendra [1984], Sastry [1984], Narendra, Annaswamy, & Singh [1985], for $p = \infty$. Lemma 3.6.2 is a version that is valid for $p \in [1,\infty]$, with a completely different proof (see the Appendix).

Note that if $\hat{H}$ is minimum phase and has relative degree zero, then it has a proper and stable inverse, and the converse result is true by lemma 3.6.1. If $\hat{H}$ is minimum phase, but has relative degree greater than zero, then the converse result will be true provided that additional conditions are placed on the input signal u. This is the result of lemma 3.6.2.

Lemma 3.6.2 Output/Input Lp Stability

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function. Let $p \in [1,\infty]$.

If $\hat{H}$ is minimum phase

For some $k_1, k_2 \geq 0$, for all $u, \dot{u} \in L_{pe}$ and for all $t \geq 0$

$$\| \dot{u}_t \|_p \leq k_1 \| u_t \|_p + k_2 \tag{3.6.5}$$

Then there exist $a_1, a_2 \geq 0$ such that

$$\| u_t \|_p \leq a_1 \| y_t \|_p + a_2 \tag{3.6.6}$$

for all $t \geq 0$.

Proof of Lemma 3.6.2 in Appendix.

It is also interesting to note the following equivalence, related to L_∞ norms. For all $a, b \in L_{\infty e}$

$$|a(t)| \leq k_1 \| b_t \|_\infty + k_2 \quad \text{iff} \quad \| a_t \|_\infty \leq k_1 \| b_t \|_\infty + k_2 \tag{3.6.7}$$

The same is true if the right-hand side of the inequalities is replaced by any positive, monotonically increasing function of time. Therefore, for $p = \infty$, the assumption (3.6.5) of lemma 3.6.2 is that u is regular (cf. definition in (2.4.14)). In particular, lemma 3.6.2 shows that if u is regular and y is bounded, then u is bounded. Lemma 3.6.2 therefore leads to the following corollary.

Corollary 3.6.3 Properties of Regular Signals

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function. Let $\hat{H}$ be stable and minimum phase and $y \in L_{\infty e}$.

(a) *If* u is regular
Then $|u(t)| \leq a_1 \| y_t \|_\infty + a_2$ for all $t \geq 0$.

(b) *If* u is bounded and $\hat{H}$ is strictly proper
Then y is regular.

(c) *If* u is regular
Then y is regular.

The properties are also valid if u and y are vectors such that each component y_i of y is related to the corresponding u_i through $y_i = \hat{H}(u_i)$.

Proof of Corollary 3.6.3 in Appendix.

In Chapter 2, a key property of the identification algorithms was obtained in terms of a gain belonging to L_2. Lemma 3.6.4 is useful for such gains appearing in connection with systems with rational transfer function $\hat{H}$.

Lemma 3.6.4

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, rational transfer function.

If $\hat{H}$ is stable, $u \in L_{\infty e}$, and for some $x \in L_{\infty e}$

$$|u(t)| \leq \beta_1(t) \| x_t \|_\infty + \beta_2(t) \tag{3.6.8}$$

for all $t \geq 0$ and for some $\beta_1, \beta_2 \in L_2$

Then there exist $\gamma_1, \gamma_2 \in L_2$ such that, for all $t \geq 0$

$$|y(t)| \leq \gamma_1(t) \| x_t \|_\infty + \gamma_2(t) \tag{3.6.9}$$

If in addition, either $\hat{H}$ is strictly proper,
or $\beta_1, \beta_2 \in L_\infty$ and $\beta_1(t), \beta_2(t) \to 0$ as $t \to \infty$

Then $\gamma_1, \gamma_2 \in L_\infty$ and $\gamma_1(t), \gamma_2(t) \to 0$ as $t \to \infty$

Proof of Lemma 3.6.4 in Appendix.

The following lemma is the so-called *swapping lemma* (Morse [1980]), and is essential to the stability proofs presented in Section 3.7.

Lemma 3.6.5 Swapping Lemma

Let $\phi, w: \mathbb{R}_+ \to \mathbb{R}^n$ and ϕ be differentiable. Let $\hat{H}$ be a proper, rational transfer function.

If $\hat{H}$ is stable, with a minimal realization

$$\hat{H} = c^T (sI - A)^{-1} b + d \tag{3.6.10}$$

Then

$$\hat{H}(w^T \phi) - \hat{H}(w^T)\phi = \hat{H}_c(\hat{H}_b(w^T)\dot{\phi}) \tag{3.6.11}$$

where

$$\hat{H}_b = (sI - A)^{-1} b \qquad \hat{H}_c = -c^T (sI - A)^{-1} \tag{3.6.12}$$

Proof of Lemma 3.6.5 in Appendix.

Lemma 3.6.6 is the so-called *small gain theorem* (Desoer & Vidyasagar [1975]) and concerns general nonlinear time-varying systems connected as shown in Figure 3.9.

Roughly speaking, the small gain theorem states that the system of Figure 3.9, with inputs u_1, u_2 and outputs y_1, y_2, is BIBO stable, provided that H_1 and H_2 are BIBO stable and provided that the product of the gains of H_1 and H_2 is less than 1.

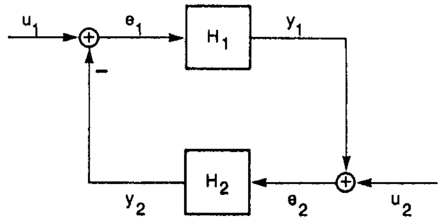

Figure 3.9: Feedback System for the Small-Gain Theorem

Lemma 3.6.6 Small Gain Theorem

Consider the system shown in Figure 3.9. Let $p \in [1, \infty]$. Let H_1, $H_2 : L_{pe} \to L_{pe}$ be causal operators. Let $e_1, e_2 \in L_{pe}$ and define u_1, u_2 by

$$u_1 = e_1 + H_2(e_2)$$

$$u_2 = e_2 - H_1(e_1) \qquad (3.6.13)$$

Suppose that there exist constants β_1, β_2 and $\gamma_1, \gamma_2 \geq 0$, such that

$$\| H_1(e_1)_t \| \leq \gamma_1 \| e_{1_t} \| + \beta_1$$

$$\| H_2(e_2)_t \| \leq \gamma_2 \| e_{2_t} \| + \beta_2 \qquad \text{for all } t \geq 0 \qquad (3.6.14)$$

If $\gamma_1 \cdot \gamma_2 < 1$

Then

$$\| e_{1_t} \| \leq (1 - \gamma_1\gamma_2)^{-1} (\| u_{1_t} \| + \gamma_2 \| u_{2_t} \| + \beta_2 + \gamma_2\beta_1)$$

$$\| e_{2_t} \| \leq (1 - \gamma_1\gamma_2)^{-1} (\| u_{2_t} \| + \gamma_1 \| u_{1_t} \| + \beta_1 + \gamma_1\beta_2) \quad (3.6.15)$$

for all $t \geq 0$.

If in addition, $u_1, u_2 \in L_p$

Then $e_1, e_2, y_1 = H_1(e_1), y_1 = H_2(e_2) \in L_p$ and (3.6.15) is valid with all subscripts t dropped.

Proof of Lemma 3.6.6 cf. Desoer & Vidyasagar [1975], p. 41.

3.7 STABILITY PROOFS

3.7.1 Stability—Input Error Direct Adaptive Control

The following theorem is the main stability theorem for the input error direct adaptive control scheme. It shows that, given any initial condition and any bounded input $r(t)$, the states of the adaptive system remain bounded (BIBS stability) and the output error tends to zero, as $t \to \infty$. Further, the error is bounded by an L_2 function.

We also obtain that the difference between the regressor vector v and the corresponding model vector v_m tends to zero as $t \to \infty$ and is in L_2. This result will be useful to prove exponential convergence in Section 3.8.

We insist that initial conditions must be in some small B_h, because although the properties are valid for any initial conditions, the convergence of the error to zero and the L_2 bounds are not uniform globally. For example, there does not exist a fixed L_2 function that bounds the output error no matter how large the initial conditions are.

Theorem 3.7.1

Consider the input error direct adaptive control scheme described in Section 3.3.1, with initial conditions in an arbitrary B_h.

Then

(a) all states of the adaptive system are bounded functions of time.

(b) the output error $e_0 = y_p - y_m \in L_2$ and tends to zero as $t \to \infty$; the regressor error $v - v_m \in L_2$ and tends to zero as $t \to \infty$.

Comments

The proof of the theorem is organized to highlight the main steps that we described in Section 3.4.

Although the theorem concerns the adaptive scheme with the gradient algorithm, examination of the proof shows that it only requires the standard identifier properties resulting from theorems 2.4.1–2.4.4. Therefore, theorem 3.7.1 is also valid if the normalized gradient algorithm is replaced by the normalized least-squares (LS) algorithm with covariance resetting.

Proof of Theorem 3.7.1

(a) *Derive properties of the identifier that are independent of the boundedness of the regressor—Existence of the solutions.*

Properties obtained in theorems 2.4.1–2.4.4 led to

$$|\phi^T(t)v(t)| = |\beta(t)|\,\| v_t \|_\infty + |\beta(t)|$$

$$\beta \in L_2 \cap L_\infty$$

$$\phi \in L_\infty \qquad , \qquad \dot{\phi} \in L_2 \cap L_\infty$$

$$c_0(t) \geq c_{\min} > 0 \qquad \text{for all } t \geq 0 \tag{3.7.1}$$

The inequality for $c_0(t)$ follows from the use of the projection in the update law.

The question of the *existence* of the solutions of the differential equations describing the adaptive system may be answered as follows. The proof of (3.7.1) indicates that $\phi \in L_\infty$ as long as the solutions exist. In fact, $|\phi(t)| \leq |\phi(0)|$, so that $|\theta(t)| \leq \theta^* + |\phi(0)|$ for all $t \geq 0$. Therefore, the controlled system is a linear time invariant system with a linear time varying controller and *bounded* feedback gains. From proposition 1.4.1, it follows that all signals in the feedback loop, and therefore in the whole adaptive system, belong to $L_{\infty e}$.

(b) *Express the system states and inputs in term of the control error.*

This was done in Section 3.5 and led to the control error $\phi^T w$, with

$$r_p = r + \frac{1}{c_0^*}\phi^T w$$

$$u = \hat{P}^{-1}\hat{M}(r_p)$$

$$y_p = \hat{M}(r_p) = y_m + \frac{1}{c_0^*}\hat{M}(\phi^T w)$$

$$\bar{w} = \begin{bmatrix} (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1}\hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix} (r_p) = \hat{H}_{\bar{w}_m r}(r_p)$$

$$= \bar{w}_m + \hat{H}_{\bar{w}_m r}\left(\frac{1}{c_0^*}\phi^T w\right) \tag{3.7.2}$$

where the transfer functions $\hat{M}$ and $\hat{H}_{\bar{w}_m r}$ are stable and strictly proper.

(c) *Relate the identifier error to the control error.*

The properties of the identifier are stated in terms of the error $\phi^T v = \phi^T L^{-1}(z)$, while the control error is $\phi^T w$. The relationship between the two can be examined in two steps.

(c1) Relate $\phi^T w$ to $\phi^T z$

Only the first component of w, namely r, is different from the first component of z, namely, r_p. The two can be related using (3.5.4), that is

$$u = c_0^* r_p + \bar{\theta}^{*T}\bar{w} \tag{3.7.3}$$

and using the fact that the control input

$$u = c_0 r + \bar{\theta}^T \bar{w} \tag{3.7.4}$$

to obtain

$$r_p = \frac{1}{c_0^*}(c_0 r + \bar{\phi}^T \bar{w}) = r + \frac{1}{c_0^*}\phi^T w \tag{3.7.5}$$

and

$$r = \frac{1}{c_0}(c_0^* r_p - \bar{\phi}^T \bar{w}) = r_p - \frac{1}{c_0}\phi^T z \tag{3.7.6}$$

It follows that

$$\frac{1}{c_0^*}\phi^T w = \frac{1}{c_0}\phi^T z \tag{3.7.7}$$

(c2) Relate $\phi^T z$ to $\phi^T v = \phi^T \hat{L}^{-1}(z)$

This relationship is obtained through the swapping lemma (lemma 3.6.5). We have, with notation borrowed from the lemma

$$\hat{L}^{-1}\left(\frac{1}{c_0}\phi^T z\right) = \frac{1}{c_0}\phi^T v + \hat{L}_c^{-1}\left(\hat{L}_b^{-1}(z^T)\left(\frac{\dot{\phi}}{c_0}\right)\right) \tag{3.7.8}$$

and, using (3.7.7) with (3.7.8)

$$\frac{1}{c_0^*}\hat{M}(\phi^T w) = \hat{M}\hat{L}\left(\hat{L}^{-1}\left(\frac{1}{c_0^*}\phi^T w\right)\right) = \hat{M}\hat{L}\left(\hat{L}^{-1}\left(\frac{1}{c_0}\phi^T z\right)\right)$$

$$= \hat{M}\hat{L}\left(\frac{1}{c_0}\phi^T v\right) + \hat{M}\hat{L}\hat{L}_c^{-1}\left(\hat{L}_b^{-1}(z^T)\left(\frac{\dot{\phi}}{c_0}\right)\right) \tag{3.7.9}$$

With (3.7.2), this equation leads to Figure 3.10. It represents the plant as the model transfer function with the control error $\phi^T w$ in feedback. The control error has now been expressed as a function of the identifier error $\phi^T v$ using (3.7.9).

The gain ϕ^T operating on v is equal to the gain β operating on $\| v_t \|_\infty$, and this gain belongs to L_2. On the other hand, $\dot{\phi} \in L_2$, so that any of its component is in L_2. In particular $\dot{c}_0 \in L_2$. Also, $c_0(t) \geq c_{\min}$, so that $1/c_0 \in L_\infty$. Thus, $d/dt(\phi/c_0) \in L_2$. Therefore, in Figure 3.10, the controlled plant appears as a stable transfer function $\hat{M}$ with an L_2 feedback gain.

(d) *Establish the regularity of the signals.*

The need to establish the regularity of the signals can be understood from the following. We are not only concerned with the boundedness of

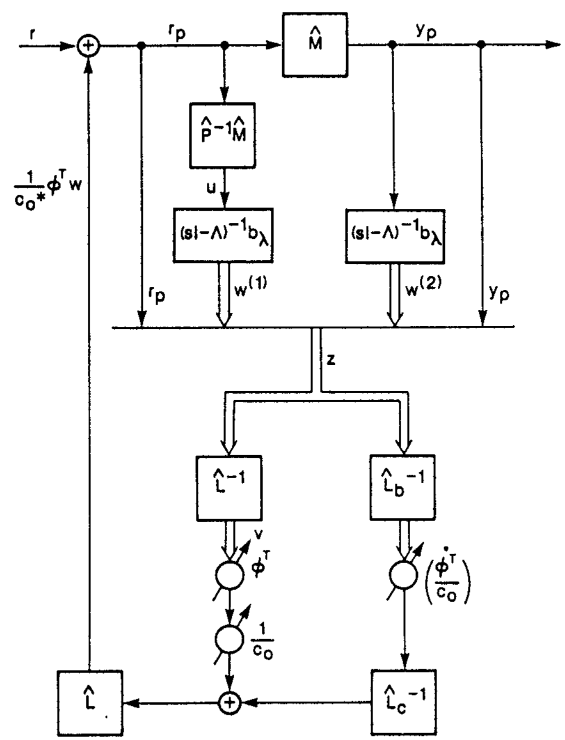

Figure 3.10: Representation of the Plant for the Stability Analysis

the output y_p but also of all the other signals present in the adaptive system. By ensuring the regularity of the signals in the loop, we guarantee, using lemma 3.6.2, that boundedness of one signal implies boundedness of all the others.

Now, note that since $\phi \in L_\infty$, the controller parameter θ is also bounded. It follows, from proposition 1.4.1, that all signals belong to $L_{\infty e}$.

Recall from (3.7.5) that

$$r_p = \frac{c_0}{c_0^*} r + \frac{1}{c_0^*} \bar{\phi}^T \bar{w} \tag{3.7.10}$$

Note that c_0 and r are bounded, by the results of (a) and by assumption (A3). $\bar{w}$ is related to r_p through a strictly proper, stable transfer function

(cf. (3.7.2)). Therefore, with (3.7.10) and lemma 3.6.1

$$|\bar{w}| \le k \| (\bar{\phi}^T \bar{w})_t \|_\infty + k$$

$$\left| \frac{d}{dt} \bar{w} \right| \le k \| (\bar{\phi}^T \bar{w})_t \|_\infty + k \tag{3.7.11}$$

for some constant $k \ge 0$. To prevent proliferation of constants, we will hereafter use the single symbol k, whenever such an inequality is valid for some positive constant.

Since ϕ is bounded, the last inequality implies that

$$\left| \frac{d}{dt} \bar{w} \right| \le k \| \bar{w}_t \|_\infty + k \tag{3.7.12}$$

that is, that $\bar{w}$ is regular.

Similarly, since ϕ and $\dot{\phi}$ are bounded and using (3.7.11)

$$\left| \frac{d}{dt} (\bar{\phi}^T \bar{w}) \right| \le \left| \left(\frac{d}{dt} \bar{\phi}^T\right) \bar{w} \right| + \left| \bar{\phi}^T \left(\frac{d}{dt} \bar{w}\right) \right|$$

$$\le k \| (\bar{\phi}^T \bar{w})_t \|_\infty + k \tag{3.7.13}$$

so that $\bar{\phi}^T \bar{w}$ is also regular.

The output y_p is given by (using (3.7.10))

$$y_p = \hat{M}(r_p) = \frac{1}{c_0^*} \hat{M}(c_0 r) + \frac{1}{c_0^*} \hat{M}(\bar{\phi}^T \bar{w}) \tag{3.7.14}$$

where $\hat{M}(c_0 r)$ is bounded. Using lemma 3.6.2, with the fact that $\bar{\phi}^T \bar{w}$ is regular and then (3.7.14)

$$|\bar{\phi}^T \bar{w}| \le k \| (\hat{M}(\bar{\phi}^T \bar{w}))_t \|_\infty + k$$

$$\le k \| y_{p_t} \|_\infty + k \| (\hat{M}(c_0 r))_t \|_\infty + k$$

$$\le k \| y_{p_t} \|_\infty + k \tag{3.7.15}$$

hence, with (3.7.10) and (3.7.11)

$$|r_p| \le k \| (\bar{\phi}^T \bar{w})_t \|_\infty + k \le k \| y_{p_t} \|_\infty + k$$

$$|\dot{\bar{w}}| \le k \| y_{p_t} \|_\infty + k \tag{3.7.16}$$

Inequalities in (3.7.16) show that the boundedness of y_p implies the boundedness of $r_p, \bar{w}, u, \ldots$, and therefore of all the states of the adaptive system.

It also follows that v is regular, since it is the sum of two regular signals, specifically

$$v = \hat{L}^{-1}(z) = \begin{bmatrix} \hat{L}^{-1} r_p \\ \hat{L}^{-1} \overline{w} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{L}^{-1} \dfrac{c_0}{c_0^*} r \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{L}^{-1} \dfrac{1}{c_0^*} \overline{\phi}^T \overline{w} \\ \hat{L}^{-1} \overline{w} \end{bmatrix} \qquad (3.7.17)$$

where the first term is the output of $\hat{L}^{-1}$ (a stable and strictly proper, minimum phase LTI system) with bounded input, while the second term is the output of $\hat{L}^{-1}$ with a regular input (cf. corollary 3.6.3).

(e) *Stability proof.*

Since v is regular, theorem 2.4.6 shows that $\beta \to 0$ as $t \to \infty$. From (3.7.2) and (3.7.9)

$$y_p = y_m + \frac{1}{c_0^*} \hat{M}\,(\phi^T w)$$

$$= y_m + \hat{M}\,\hat{L}\,(\frac{1}{c_0}\,\phi^T v) + \hat{M}\,\hat{L}\,\hat{L}_c^{-1}(\hat{L}_b^{-1}(z^T)(\frac{\dot{\phi}}{c_0})) \qquad (3.7.18)$$

We will now use the single symbol β in inequalities satisfied for some function satisfying the same conditions as β, that is $\beta \in L_2 \cap L_\infty$ and $\beta(t) \to 0$, as $t \to \infty$.

The transfer functions $\hat{M}\,\hat{L}$, $\hat{L}_b^{-1}$ and $\hat{L}_c^{-1}$, are all stable and the last two are strictly proper. The gain $\dfrac{1}{c_0}$ is bounded by (3.7.2), because of the projection in the update law. Therefore, using results obtained so far and lemmas 3.6.1 and 3.6.4

$$\begin{aligned} |y_p - y_m| &\le \beta \| v_t \|_\infty + \beta \| z_t \|_\infty + \beta \\ &\le \beta \| r_{p_t} \|_\infty + \beta \| \overline{w}_t \|_\infty + \beta \\ &\le \beta \| y_{p_t} \|_\infty + \beta \\ &\le \beta \| (y_p - y_m)_t \|_\infty + \beta \qquad (3.7.19) \end{aligned}$$

Recall that since $\theta \in L_\infty$, all signals in the adaptive system belong to $L_{\infty e}$. On the other hand, for T sufficiently large, $\beta(t \ge T) < 1$.

Therefore, application of the small gain theorem (lemma 3.6.6) with (3.7.19) shows that $y_p - y_m$ is bounded for $t \ge T$. But since y_p, $y_m \in L_{\infty e}$, it follows that $y_p \in L_\infty$. Consequently, all signals belong to L_∞.

From (3.7.19), it also follows that $e_0 = y_p - y_m \in L_2$, and tends to zero as $t \to \infty$. Similarly, using (3.5.9), (3.5.12) and (3.7.9)

$$v = v_m + \begin{bmatrix} 1 \\ (sI - \Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M} \\ \hat{M} \\ (sI - \Lambda)^{-1} b_\lambda \hat{M} \end{bmatrix}$$

$$(\frac{1}{c_0}\,\phi^T v + \hat{L}_c^{-1}(\hat{L}_b^{-1}(z^T)(\frac{\dot{\phi}}{c_0}))) \qquad (3.7.20)$$

so that $v - v_m$ also belongs to L_2 and tends to zero, as $t \to \infty$.

3.7.2 Stability—Output Error Direct Adaptive Control

Theorem 3.7.2

Consider the output error direct adaptive control scheme described in Section 3.3.2, with initial conditions in an arbitrary B_h.

Then

(a) all states of the adaptive system are bounded functions of time.

(b) the output error $e_0 = y_p - y_m \in L_2$ and tends to zero as $t \to \infty$ the regressor error $\hat{L}^{-1}(w) - \hat{L}^{-1}(w_m) \in L_2$ and tends to zero as $t \to \infty$.

Proof of Theorem 3.7.2

The proof is very similar to the proof for the input error scheme, and is just sketched here, following the steps of the proof of theorem 3.7.1.

(a) We now have, instead

$$e_1, \dot{\overline{\phi}} \in L_2$$

$$e_1, \overline{\phi} \in L_\infty \qquad (3.7.21)$$

Note that these results are valid, although the realization of $\hat{M}$ is not minimal (but is stable).

(b) As in theorem 3.7.1.

(c) Since $c_0 = c_0^*$, (3.7.9) becomes

$$\frac{1}{c_0^*}\hat{M}(\bar{\phi}^T\bar{w})$$

$$= \frac{1}{c_0^*}\hat{M}\hat{L}(\bar{\phi}^T\bar{v}) + \frac{1}{c_0^*}\hat{M}\hat{L}(\hat{L}_c^{-1}(\hat{L}_b^{-1}(\bar{w}^T)\dot{\bar{\phi}})) \tag{3.7.22}$$

(d) As in theorem 3.7.1, it follows that $\bar{w}$ is regular—cf. (3.7.12). Unfortunately, $\bar{\phi}^T\bar{w}$ is not necessarily regular because $\dot{\bar{\phi}}$ is not bounded. However, we will show that (3.7.16) still holds, which is all we will need for the stability proof.

To prove (3.7.16), first note that lemma 3.6.2 may be modified as follows.

If there exists $z \in L_{pe}$, $k_3 \geq 0$ such that (3.6.5) is replaced by

$$\|\dot{u}_t\|_p \leq k_1 \|u_t\|_p + k_2 + k_3\|z_t\|_p \tag{3.7.23}$$

then lemma 3.6.2 is valid with (3.6.6) replaced by

$$\|u_t\|_p \leq a_1 \|y_t\|_p + a_2 + a_3\|z_t\|_p \tag{3.7.24}$$

for some $a_3 \geq 0$. We leave it to the reader to verify this new version of the lemma.

Now, recall that

$$\bar{w} = \begin{bmatrix} w^{(1)} \\ y_p \\ w^{(2)} \end{bmatrix} = \begin{bmatrix} (sI-\Lambda)^{-1} b_\lambda(u) \\ y_p \\ (sI-\Lambda)^{-1} b_\lambda(y_p) \end{bmatrix} \tag{3.7.25}$$

so that

$$\|w_t^{(2)}\| \leq k\|y_{p_t}\| + k \tag{3.7.26}$$

To apply the modified lemma 3.6.2, we note from (3.7.25) that

$$w^{(2)} = \hat{P}(w^{(1)}) \tag{3.7.27}$$

where $\hat{P}$ is minimum phase. Further

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u = \Lambda w^{(1)} + b_\lambda c_0^* r + b_\lambda \bar{\theta}^T\bar{w} \tag{3.7.28}$$

It follows that

$$\|\dot{w}^{(1)}\|_\infty \leq k\|w_t^{(1)}\|_\infty + k + k(\|w_t^{(2)}\|_\infty + \|y_{p_t}\|_\infty) \tag{3.7.29}$$

and, applying the modified lemma 3.6.2

$$\|w_t^{(1)}\|_\infty \leq k\|w_t^{(2)}\|_\infty + k\|y_{p_t}\|_\infty + k \tag{3.7.30}$$

Putting (3.7.30) with (3.7.25)–(3.7.26)

$$\|\bar{w}_t\|_\infty \leq \|w_t^{(1)}\|_\infty + \|y_{p_t}\|_\infty + \|w_t^{(2)}\|_\infty$$

$$\leq k\|y_{p_t}\|_\infty + k \tag{3.7.31}$$

which is equivalent to (3.7.16) (cf. (3.6.7)).

(e) Recall, from (3.3.16) and the definition of the gradient update law, that

$$\frac{1}{c_0^*}\hat{M}\hat{L}(\bar{\phi}^T\bar{v}) = e_1 + \frac{\gamma}{c_0^*}\hat{M}\hat{L}(\bar{v}^T\bar{v}e_1)$$

$$= e_1 - \frac{\gamma}{gc_0^*}\hat{M}\hat{L}(\dot{\bar{\phi}}^T\bar{v}) \tag{3.7.32}$$

so that, with (3.7.22)

$$y_p - y_m = \frac{1}{c_0^*}\hat{M}\hat{L}(\bar{\phi}^T\bar{v}) + \frac{1}{c_0^*}\hat{M}\hat{L}(\hat{L}_c^{-1}(\hat{L}_b^{-1}(\bar{w}^T))\dot{\bar{\phi}}))$$

$$= e_1 - \frac{\gamma}{gc_0^*}\hat{M}\hat{L}(\dot{\bar{\phi}}\bar{v}) + \frac{1}{c_0^*}\hat{M}\hat{L}(\hat{L}_c^{-1}(\hat{L}_b^{-1}(\bar{w}^T)\dot{\bar{\phi}}) \tag{3.7.33}$$

Recall that e_1 is bounded (part (a)) and that $\hat{M}\hat{L}$ is strictly proper (in the output error scheme). The proof can then be completed as in theorem 3.7.1. □

3.7.3 Stability—Indirect Adaptive Control

Theorem 3.7.3

Consider the indirect adaptive control scheme described in Section 3.3.3, with initial conditions in an arbitrary B_h.

Then

(a) all states of the adaptive system are bounded functions of time.

(b) the output error $e_0 = y_p - y_m \in L_2$, and tends to zero as $t \to \infty$
the regressor error $\tilde{w} - \tilde{w}_m \in L_2$, and tends to zero as $t \to \infty$.

Proof of Theorem 3.7.3 in Appendix.

Comments

Compared with previous proofs, the proof of theorem 3.7.3 presents additional complexities due to the transformation $\pi \to \theta$. A major step is to relate the identification error $\psi^T \tilde{w}$ to the control error $\phi^T w$. We now discuss the basic ideas of the proof. The exact formulation is left to the Appendix.

To understand the approach of the proof, assume that the parameters π and θ are fixed in time and that k_p is known. For simplicity, let $k_p = a_{m+1} = k_m = 1$. The nominal values of the identifier parameters are then given by

$$\hat{a}^* = \hat{n}_p$$

$$\hat{b}^* = \hat{\lambda} - \hat{d}_p$$

The controller parameters are given as a function of the identifier parameters through

$$\hat{c} = \hat{\lambda} - \hat{q}\hat{a}$$

$$\hat{d} = \hat{q}\hat{\lambda} - \hat{q}\hat{b} - \hat{\lambda}_0 \hat{d}_m \qquad (3.7.34)$$

while the nominal values are given by

$$\hat{c}^* = \hat{\lambda} - \hat{q}^*\hat{a}^* = \hat{\lambda} - \hat{q}^*\hat{n}_p$$

$$\hat{d}^* = \hat{q}^*\hat{\lambda} - \hat{q}^*\hat{b}^* - \hat{\lambda}_0\hat{d}_m = \hat{q}^*\hat{d}_p - \hat{\lambda}_0\hat{d}_m \qquad (3.7.35)$$

It follows that

$$\hat{q}\hat{a} - \hat{q}\hat{a}^* = (\hat{\lambda} - \hat{c}) - \hat{q}\hat{n}_p = -(\hat{c} - \hat{c}^*) + (\hat{\lambda} - \hat{c}^*) - \hat{q}\hat{n}_p$$

$$= -(\hat{c} - \hat{c}^*) + (\hat{q}^* - \hat{q})\hat{n}_p \qquad (3.7.36)$$

and

$$\hat{q}\hat{b} - \hat{q}\hat{b}^* = \hat{q}\hat{\lambda} - \hat{d} - \hat{\lambda}_0\hat{d}_m - \hat{q}\hat{\lambda} + \hat{q}\hat{d}_p$$

$$= -(\hat{d} - \hat{d}^*) + (-\hat{d}^* - \hat{\lambda}_0\hat{d}_m + \hat{q}\hat{d}_p)$$

$$= -(\hat{d} - \hat{d}^*) + (\hat{q} - \hat{q}^*)\hat{d}_p \qquad (3.7.37)$$

Therefore

$$\hat{q}\left[\frac{\hat{a} - \hat{a}^*}{\hat{\lambda}} + \frac{\hat{b} - \hat{b}^*}{\hat{\lambda}}\frac{\hat{n}_p}{\hat{d}_p}\right] = -\left[\frac{\hat{c} - \hat{c}^*}{\hat{\lambda}} + \frac{\hat{d} - \hat{d}^*}{\hat{\lambda}}\frac{\hat{n}_p}{\hat{d}_p}\right] \qquad (3.7.38)$$

This equality of polynomial ratios can be interpreted as an operator equality in the Laplace transform domain, since we assumed that the parameters were fixed in time. If we apply the operator equality to the input u, it leads to (with the definitions of Section 3.3)

$$\hat{q}(\psi^T \tilde{w}) = -\bar{\phi}^T \bar{w} \qquad (3.7.39)$$

and, consequently

$$y_p - y_m = -\frac{1}{c_0^*}\hat{M}\hat{q}(\psi^T \tilde{w}) \qquad (3.7.40)$$

Since the degree of $\hat{q}$ is at most equal to the relative degree of the plant, the transfer function $\hat{M}\hat{q}$ is proper and stable. The techniques used in the proof of theorem 3.7.1 and the properties of the identifier would then lead to a stability proof.

Two difficulties arise when using this approach to prove the stability of the indirect adaptive system. The first is related to the unknown high-frequency gain, but only requires more complex manipulations. The real difficulty comes from the fact that the polynomials $\hat{q}$, $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ vary as functions of time. Equation (3.7.38) is still valid as a polynomial equality, but transforming it to an operator equality leading to (3.7.40) requires some care.

To make sense of time varying polynomials as operators in the Laplace transform domain, we define

$$\hat{s}_n = \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix} \qquad (3.7.41)$$

so that

$$\hat{a}(s) = a^T \hat{s}_n \qquad \frac{\hat{a}(s)}{\hat{\lambda}(s)} = a^T\left[\frac{\hat{s}_n}{\hat{\lambda}}\right] \qquad (3.7.42)$$

Consider the following equality of polynomial ratios

$$\frac{\hat{a}(s)}{\hat{\lambda}(s)} = \frac{\hat{b}(s)}{\hat{\lambda}(s)} \qquad (3.7.43)$$

where $\hat{a}$ and $\hat{b}$ vary with time but $\hat{\lambda}$ is a constant polynomial. Equality (3.7.43) implies the following operator equality

$$a^T\left[\frac{\hat{s}_n}{\hat{\lambda}}(.)\right] = b^T\left[\frac{\hat{s}_n}{\hat{\lambda}}(.)\right] \tag{3.7.44}$$

Similarly, consider the product

$$\frac{\hat{a}(s)}{\hat{\lambda}(s)} \cdot \frac{\hat{b}(s)}{\hat{\lambda}(s)} \tag{3.7.45}$$

This can be interpreted as an operator by multiplying the coefficients of the polynomials to lead to a ratio of higher order polynomials and then interpreting it as previously. We note that the product of polynomials can be expressed as

$$\hat{a}(s)\hat{b}(s) = a^T(\hat{s}_n\hat{s}_n^T)b \tag{3.7.46}$$

so that the operator corresponding to (3.7.45) is

$$a^T\left[\frac{\hat{s}_n}{\hat{\lambda}} \cdot \frac{\hat{s}_n^T}{\hat{\lambda}}(.)\right] b \tag{3.7.47}$$

i.e. by first operating the matrix transfer function on the argument and then multiplying by a and b in the time domain. Note that this operator is different from the operator

$$a^T\left[\frac{\hat{s}_n}{\hat{\lambda}}\left[\frac{\hat{s}_n^T}{\hat{\lambda}}(.)\,b\right]\right] \tag{3.7.48}$$

but the two operators can be related using the swapping lemma (lemma 3.6.5).

3.8 EXPONENTIAL PARAMETER CONVERGENCE

Exponential convergence of the identification algorithms under persistency of excitation conditions was established in Sections 2.5 and 2.6. Consider now the input error direct adaptive control scheme of Section 3.3.1. Using theorem 2.5.3, it would be straightforward to show that the parameters of the adaptive system converge exponentially to their nominal values, provided that the regressor v is persistently exciting. However, such result is useless, since the signal v is generated inside the adaptive system and is unknown *a priori*. Theorem 3.8.1 shows that it is sufficient for the *model* signal w_m to be persistently exciting to guarantee exponential convergence.

Note that in the case of adaptive control, we are not only interested in the convergence of the parameter error to zero, but also in the convergence of the errors between plant states and model states. In other

words, we are concerned with the exponential stability of the overall adaptive system.

Theorem 3.8.1

Consider the input error direct adaptive control scheme of Section 3.3.1.

If w_m is PE

Then the adaptive system is exponentially stable in any closed ball.

Proof of Theorem 3.8.1

Since w_m, $\dot{w}_m$ are bounded, lemma 2.6.6 implies that $v_m = \hat{L}^{-1}(z_m) = \hat{L}^{-1}(w_m)$ is PE. In theorem 3.7.1, we found that $v - v_m \in L_2$. Therefore, using lemma 2.6.5, v_m PE implies that v is PE. Finally, since v is PE, by theorem 2.5.3, the parameter error ϕ converges exponentially to zero.

Recall that in Section 3.5, it was established that the errors between the plant and the model signals are the outputs of stable transfer functions with input $\phi^T w$. Since w is bounded (by theorem 3.7.1), $\phi^T w$ converges exponentially to zero. Therefore, all errors between plant and model signals converge to zero exponentially fast. □

Comments

Although theorem 3.8.1 establishes exponential stability in any closed ball, it does not prove global exponential stability. This is because $v - v_m$ is not bounded by a unique L_2 function for any initial condition. Results in Section 4.5 will actually show that the adaptive control system is not globally exponentially stable.

The various theorems and lemmas used to prove theorem 3.8.1 can be used to obtain estimates of the convergence rates of the parameter error. It is, however, doubtful that these estimates would be of any practical use, due to their complexity and to their conservatism. A more successful approach is that of Chapter 4, using averaging techniques.

The result of theorem 3.8.1 has direct parallels for the other adaptive control algorithms presented in Section 3.3.

Theorem 3.8.2

Consider the output error direct adaptive control scheme of Section 3.3.2 (or the indirect scheme of Section 3.3.3).

If w_m is PE ($\tilde{w}_m$ is PE)

Then the adaptive system is exponentially stable in any closed ball.

Proof of Theorem 3.8.2

The proof of theorem 3.8.2 is completely analogous to the proof of theorem 3.8.1 and is omitted here. □

3.9 CONCLUSIONS

In this chapter, we derived three model reference adaptive control schemes. All had a similar controller structure but different identification structures. The first two schemes were direct adaptive control schemes, where the parameters updated by the identifier were the same as those used by the controller. The third scheme was an indirect scheme, where the parameters updated by the identifier were the same as those of the basic identifier of Chapter 2. Then, the controller parameters were obtained from the identifier parameters through a nonlinear transformation resulting from the model reference control objective.

We investigated the connections between the adaptive control schemes and also with other known schemes. The difficulties related to the unknown high-frequency gain were also discussed. The stability of the model reference adaptive control schemes was proved, together with the result that the error between the plant and the reference model converged to zero as t approached infinity. We used a unified framework and an identical step-by-step procedure for all three schemes. We proved basic lemmas that are fundamental to the stability proofs and we emphasized a basic intuitive idea of the proof of stability, that was the existence of a *small loop gain* appearing in the adaptive system.

The exponential parameter convergence was established, with the additional assumption of the persistency of excitation of a model regressor vector. This condition was to be satisfied by an exogeneous model signal, influenced by the designer and was basically a condition on the reference input.

An interesting conclusion is that the stability and convergence properties are identical for all three adaptive control schemes. In particular, the indirect scheme had the same stability properties as the direct schemes. Further, the normalized gradient identification algorithm can be replaced by the least squares algorithm with projection without altering the results. Differences appear between the schemes however, in connection with the high-frequency gain and with other practical considerations.

The input error direct adaptive control scheme and the indirect scheme are attractive because they lead to linear error equations and do not involve SPR conditions. Another advantage is that they allow for a decoupling of identification and control useful in practice. The indirect scheme is quite more intuitive than the input error direct scheme,

although more complex in implementation and especially as far as the analysis is concerned. The final result however shows that stability is not an argument to prefer one over the other.

The various model reference adaptive control schemes also showed that the model reference approach is not bound to the choice of a direct adaptive control scheme, to the use of the output error in the identification algorithm, or to SPR conditions on the reference model.